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VIBRATION IN ENGINEERING

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PREFACE

THE various phenomena dealt with in this book have hitherto scarcely received the attention which they deserve.

The questions raised are all of vital importance to the manufacturer and user of machinery, as well as to the student of engineering.

Unhappily the subject is generally brought to one's notice by some form of trouble, and it is then found that the information and literature on the subject are extremely scattered, and very difficult to co-ordinate. The reason for this is that the right understanding of the subject involves diverse problems in sound, strength of materials and mechanics, as well as the physics and mathematics of harmonic motion.

We have endeavoured in the following pages to bring together and co-ordinate these different aspects, and to present the subject of engineering vibration as a unity.

In this we have catered for the two types of engineering minds which undoubtedly exist, by presenting the subject first from the physical and then from the mathematical standpoint.

Some of the presentations are, as far as we know, new, such as the treatment of whirling shafts and of the cyclic irregularity of alternators.

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LIST OF SYMBOLS USED

PART I

		PAGE
$a =$	a displacement from position of rest in feet	6 etc.
$C =$	coefficient of rigidity of material of shaft .	13
$c =$	torsional stiffness of shaft in ft. lb. weight per radian	13
$d =$	elastic deflection of a shaft in whirl . .	17
$f =$	force to pull spring or bend shaft in lb. weight per ft. displacement . .	6, 16 etc.
$g =$	32·2	7 etc.
$H =$	the horizontal component of the earth's magnetic field	11
$I =$	second moment of area of a shaft in $(ft.)^4$.	13 etc.
$k =$	radius of gyration of a flywheel in feet .	13
$l =$	half length of a magnet pivoted at middle	11
	length in feet of a pendulum . . .	12
	length in feet of a shaft	13
	length in feet of the limb of a U-tube .	35
	length in feet of suspension wires . .	48
$M =$	mass in pounds	7 etc.
$m =$	strength of poles of a magnet. C.G.S. units	11
$N =$	number of power impulses per revolution of an engine	44
$n =$	revolutions per second	17 etc.
$R =$	actual total eccentricity of a shaft . .	26
$R_0\ R_1\ R_2 =$	mean, lowest and highest revolutions per minute of an engine	43
$r =$	initial eccentricity	17
$T =$	time of a complete vibration in seconds .	16 etc.
$V =$	maximum value of a varying velocity .	7
$v =$	circumferential velocity	24

		PAGE
$\alpha =$	2π times a frequency	26
$\theta =$	angle of lag between stator and rotor poles	39
$\phi =$	angle between disturbing force and displacement	24
$\mu =$	coefficient of friction	24
ω	angular velocity in radians per second	17

PART II

$A =$	area of section	54
	a constant	90
	half amplitude of vibration	60 etc.
$a =$	element of area	54 etc.
	half amplitude of vibration	59 etc.
	a numerical coefficient	95
	segment of a shaft	97 etc.
$B =$	a constant	90
	half amplitude of vibration	60 etc.
$b =$	a numerical coefficient	95
	segment of a shaft	97 etc.
$C =$	coefficient of rigidity	81
	a constant	90
	a constant of integration	63 etc.
$c =$	torsional stiffness of shaft in foot poundals per radian	85
$D = \dfrac{d}{dt}$		63
	a constant	90
	a constant of integration	97 etc.
$d =$	half length of bar	78
$E =$	coefficient of elasticity	80 etc.
$e =$	base of hyperbolic functions	63 etc.
	strain	80
$F =$	a force in poundals	53 etc.
$f =$	stiffness of a spring in poundals per foot of deflection	61 etc.
$f =$	stress in poundals per square foot	80 etc.

		PAGE
g =	32·2	53
h =	distance between two axes	55 etc.
I =	second moment of area	54 etc.
i =	$\sqrt{-1}$	63 etc.
	slope of a shaft	90
K =	a section on a shaft	97
k =	$\dfrac{\mu}{M}$	
	radius of gyration	55 etc.
$\bar{k}$ =	radius of gyration about a mass centre	55
L =	length in feet of a whirling shaft	91 etc.
l =	length in feet of a pendulum	76
	length in feet of a shaft	81 etc.
	length in feet of a suspension wire	78 etc.
M =	mass in pounds	53 etc.
	bending moment in foot poundals	88 etc.
m =	mass in pounds	53 etc.
	mass in pounds per foot length of shaft	90
N =	moment of inertia in pounds feet^2	85
N_A, N_B =	end fixing moments	99 etc.
O =	intersection of axes	56
P =	force in poundals	68 etc.
p =	$\dfrac{P}{M}$	69
q =	shear stress in poundals per sq. foot	81 etc.
R =	radius of curvature	88
R_1, R_2 =	inner and outer radii of an annular section	58
R_A, R_B =	reactions at bearings	97 etc.
S =	shearing force	89
s =	linear displacement	75
T =	time of vibration	59 etc.
	torque in foot poundals	81
t =	time	59 etc.
v =	velocity in feet per second	52
w =	intensity of loading in poundals per foot length	89

		PAGE
x =	distance along a shaft from a point of reference	90 etc.
	momentary value of a displacement in feet	61 etc.
y =	deflection of a shaft in feet	89 etc.
α =	angular velocity in radians per second	52
θ =	angular displacement in radians	81 etc.
μ =	coefficient of friction	62
ϕ =	shear strain	81
ω =	angular velocity in radians per second	52

VIBRATION IN ENGINEERING

PART I. PHYSICAL

(See p. xi for List of Symbols used in this part of the book.)

VIBRATION may be defined as a series of recurring excursions, either from the position of rest or from a state of uniform motion. A plucked fiddle-string, and the motion of a particle of air carrying a sound in a wind, are two examples illustrating these two types of vibration. These excursions are, in general, on both sides of the position of rest, but may sometimes be on one side only, as, for example, a ball bouncing on a hard surface.

Causes of Vibration.

From the above, it will be seen that anything which is movable can be set into a state of vibration, and any recurring impulse can be a source of vibration. These recurring impulses may take the form of some engineering operation, such as the working of a trip hammer; any revolving machinery having any part out of balance, or an engine whose turning effort is not perfectly uniform; any sound also falls into this category, as does the vibration caused by vehicles on the road's surface. Another set of causes is exemplified by the blades of a propeller or fan on water or air. Vibration may be intentionally caused by, say, drawing a fiddle-bow over a stretched string (in which case it is the result of static friction being greater than dynamic), or by a column of air being set into vibration in an organ pipe.

A body set into a state of vibration by any of these causes will vibrate with an increasing amplitude (as the total length of the excursions is called) until the resistance to such motion, usually in the form of friction, balances the disturbing force,

and a more or less uniform state of vibration continues till the exciting cause is withdrawn.

Bodies With and Without Natural Periods of Vibration.

Things which can vibrate fall into two general categories. The first is one in which the body has no tendency in itself to vibrate, and merely does so in answer to the application of the disturbing force. Bodies in the second category contain in themselves or their surroundings an inherent aptitude to vibrate. If we look into the causes of this we shall find that two properties and only two are necessary. They are, first, mass or inertia, and, secondly, the existence of a restoring force which tends to bring the body back to its position of rest on its being displaced from it. Of mass or inertia, little need be said except that it is becoming the recognized view that this property of matter is due to the presence of electric charges which it carries or of which it consists.

Examples of Restoring Forces.

There are many different sorts of restoring forces which are met with in engineering. There is first, Gravity, or that force with which all matter is attracted to all other matter. The pendulum bob, hanging on a string, if displaced from its position of rest, tends to be restored thereto by the force of gravity, and, possessing mass or inertia, the bob will vibrate under these two causes till it is again brought to rest by friction.

Another restoring force is supplied by the stiffness of material. If one end of a steel bar is clamped in a vice, and the other is displaced and let go, the requisite combination of mass and restoring force causes a state of vibration. The case of the fiddle-string has already been mentioned. The restoring force here is the stretching force at the ends of the string, and the mass is usually distributed uniformly along the length of the string. With magnetic forces the engineer is quite familiar, although he does not always fully understand the mechanism by which they work. If a compass needle is displaced from its position North and South it will vibrate under the action of its own mass and the restoring force exerted by the earth's magnetic field. In some such way a quantity of electricity

will sometimes surge to and fro in a circuit, the self-induction of which is probably more than analogous to inertia, the capacity of the circuit lending the necessary elasticity and storing effect.

Different Actions of Disturbing Forces.

It will be seen that bodies coming under these two headings will behave differently when subjected to a rhythmic disturbing force. Bodies in the first category will be moved back and forth in a greater or less degree according to their mass, or to the effectiveness with which they are held still. The frequency of the resulting vibration will be in all cases the same as that of the disturbing force, and on the withdrawal of the latter their state of vibration will cease. The effect of a disturbing force on the bodies falling into the second category, however, will be, in general, quite different. When any of these bodies are displaced and let go, there will be an inherent tendency for them to vibrate with a frequency depending only on their mass and on the magnitude of the restoring forces. If acted on by a recurring impulse, they will, like members of the former class, vibrate with the frequency of the disturbing impulses, and the resultant motion will be a compromise between that dictated by the disturbing force and that in which the body itself would naturally vibrate. If the frequency of the disturbing force is very far removed from the natural frequency of the system itself, the body will vibrate in very much the same way as if it were a member of the other category, but if, on the other hand, the frequency of the disturbing force is at all near the natural frequency of the system, then the resulting vibration will be profoundly altered, and it is in cases coming under this latter heading that the engineer is more often interested.

Beats.

If the disturbing force has a frequency nearly but not quite equal to the natural frequency of the thing vibrated we may have the phenomenon of beats, *i.e.* the amplitude of the excursion increases and diminishes again with a rhythm of longer and longer period as the two frequencies approach one another. In general, these beats begin to be dangerous when the two frequencies are within about 20 per cent. of each other.

A mass or system of masses and restoring forces may have quite different methods of vibration according to where and how the disturbing forces are applied.

It will now be helpful to investigate the laws of vibration and the sort of motion which results from different arrangements of mass and restoring force.

Units.

Before we can proceed to find the numerical relations between the various quantities involved, we must decide what system of units to use and then stick to this system. Although the C.G.S. or other absolute system of units would be very much simpler to handle from the point of view of the authors of this book, and give rise to much prettier expressions in print, we are constrained to fall back on the British engineering units, because of the difficulty which will otherwise inevitably be met with in putting into practice any of the conclusions at which we arrive.

The accepted units used in British engineering are :—

Mass.—The pound. Mass is a quantity of matter, and the standard quantity is a lump of platinum kept in the Tower of London. Mass is a fundamental unit which is quite irrespective of gravity. Other masses can be compared with the standard mass by means of either of its two main properties :—

(*a*) The acceleration it acquires when acted on by a force, or,

(*b*) The attraction which is exerted upon it by other quantities of matter, *e.g.* by the Earth.

Length.—The foot.

Time.—We generally use the Minute, but very often the Second.

Force.—The weight of a pound, *i.e.* the attraction which the earth exerts on the mass of a pound at one particular place on the earth's surface. This is the most obvious of the forces which we encounter, and forms a useful working unit. It is only available on the earth's surface, but sub-standards can be made of springs which would work correctly anywhere.

Work.—The work done in moving the weight-of-a-pound

through a foot, written "foot-pound" for short. This may be work actually done against gravity, or it may be done against other forces, such as a spring or the friction of rolling stock on the level.

Power.—The rate of doing work—the foot-pound per minute. The sub-standard of 33,000 foot-pounds per minute is called the "horse power."

Other derived units are:—

Energy.—This, in dimensions, is the same as work, and is therefore expressed in "foot-pounds." It is sometimes also expressed as "horse-power minutes," which is, however, a somewhat cumbersome expression, the idea of time being introduced only to cancel out again, as is seen by expressing it as "foot-pound-per-minute minutes." Horse-power minutes obviously equal foot-pounds, divided by 33,000.

Moment of Inertia.—This is obtained by multiplying the mass in pounds by the square of its distance in feet from the centre of turning. This distance is called the *radius of gyration*, and is generally written k, making the moment of inertia Mk^2.

Mass on the End of a Spring.

The simplest example of vibration that can be taken is that of a mass hung from one end of a helical spring (see Fig. 1). It must first be pointed out that the action of gravity, existing as it does constantly and in one direction, does not affect the results in any way. The restoring force is that of the spring alone, and the quantity of matter hung from its end would have the property of mass or inertia, and would vibrate at exactly the same rate were the circumstance of gravity entirely withdrawn. The spring, whether it is further extended or compressed, exerts a restoring force to the position

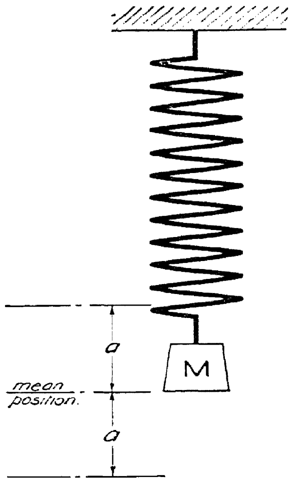


FIG. 1.

of rest which is found experimentally to be proportional to the displacement, if the spring is not locked solid or extended so as to stretch the material of which it is composed beyond its elastic limit. Gravity may perform the function of opening the spring sufficiently for our purpose; otherwise, if the experiment were made in interstellar space, the spring would have to be wound so as to be in its position of rest when slightly opened out.

Now displace the mass either upward or downward, and let it go. The mass will experience an acceleration toward the position of rest, which acceleration will equal the force at the instant exerted by the spring, divided by the mass. This force will continue, though in reducing amount, until the mass is passing through the position of rest, when the mass will have its maximum velocity, and, by reason of its inertia, will be carried beyond this position; the restoring force of the spring will then reverse and gradually bring the mass to rest by impressing on it an opposite acceleration. The process will then be repeated, reversed, exactly as before.

If the path of motion of the mass be supposed to be the diameter of a circle round the circumference of which a point is revolving with a uniform angular velocity, then the projection of this point on the diameter has exactly the same motion as the mass up and down the diameter. This fact is found to be a great convenience in the mathematical treatment of the subject of vibration. The motion on the diameter is recognised as simple harmonic motion, of which the mathematical properties are well known, and will be assumed to be known in what follows.

Let the spring be of such a strength that the force required to alter its length by one foot is f, and call the mass M. Suppose the mass is pulled down a feet from its position of rest and let go. The force acting on it at first will be $f \times a$, and will be proportional to its distance from the position of rest, at which point it will be nothing. Its average value is therefore $\frac{fa}{2}$, and the work done in the space a will be

$$\frac{fa}{2} \times a = \frac{fa^2}{2}.$$

This will be its energy when passing through the position of rest, which $= \frac{1}{2}MV^2$, V being the maximum velocity.

$$\text{Therefore} \qquad \frac{fa^2}{2} = \tfrac{1}{2}MV^2 \text{ and } V = a\sqrt{\frac{f}{M}}.$$

The actual velocity varies between zero and this maximum value, its average value being

$$V \times \frac{2}{\pi} = \frac{2}{\pi} a\sqrt{\frac{f}{M}}.$$

In one complete swing the mass moves through $4a$ feet, and therefore the time taken is equal to

$$\frac{4a}{\frac{2}{\pi} \times a\sqrt{\frac{f}{M}}} = 2\pi\sqrt{\frac{M}{f}}$$

which is of the form :

$$\text{Time of a complete vibration} = 2\pi\sqrt{\frac{\text{mass}}{\text{Restoring force per unit of displacement.}}}$$

If absolute units were used, this expression would give the correct time in seconds without any trouble, but as mass is reckoned in pounds and the restoring force in pounds weight and the displacement in feet, the result must be divided by 5·66, which is the square root of the acceleration produced by gravity at Greenwich.

This numerical complication has arisen because we have adopted the attraction of gravity on the mass of a pound as our standard of force. This attraction of gravity produces an acceleration of 32·2 feet per second per second, whereas the absolute unit of force would only produce an acceleration of 1 foot per second per second, so that our unit of force is 32·2 times bigger than the absolute unit.

$$\sqrt{g} = \sqrt{32.2} = 5{\cdot}66.$$

This apparent confusion is the result of British engineers choosing a unit of force independently, whereas the unit of force should be the logical outcome of the units of mass, length and time. It is, however, at the worst, only a matter of change of units, and can always be put right by a merely

numerical constant. The 32·2 becomes, not an acceleration, but a numerical multiplier which has no dimensions.

Measurement of Stiffness and Hardness of Materials by Vibration.

The general expression giving the relation between the moving mass and the restoring force and the time of vibration may be used to actually measure the quality of stiffness or resilience in any piece of material. Take, for example, a piece of wood which it is proposed to use in the construction of an aeroplane. In this case the material would be weighted with a definite mass, set into vibration and the frequency counted. In this way, the restoring force, or stiffness, can be calculated, all the other quantities in the expression being known. In like manner, the hardness of a sample of metal can be measured by counting the number of rebounds per minute of a steel ball dropped upon it.

Phase of Forced Vibration.

It is interesting to notice the relation between the motion caused by a recurrent disturbing force and the body vibrated by it when the body vibrated has a natural period of vibration of its own. If the frequency of the disturbing force is higher than the natural period of the body vibrated, the phase of the motion will be opposite to what one would naturally expect. This can be clearly shown by holding one end of the spring mentioned above to the other end of which is attached a mass. If the hand is moved very slowly up and down, the motion of the mass will follow it, *i.e.* will go up and down in phase with the hand. As the frequency of the hand gets nearer to the natural period of the system the vibration becomes violent, but, after passing through this stage, and moving the top of the spring much faster than the natural period, the phases are seen to have reversed, the mass going upwards

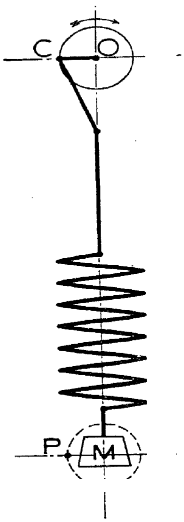


FIG. 2.

as the end of the spring is moved downwards, and *vice versa.*

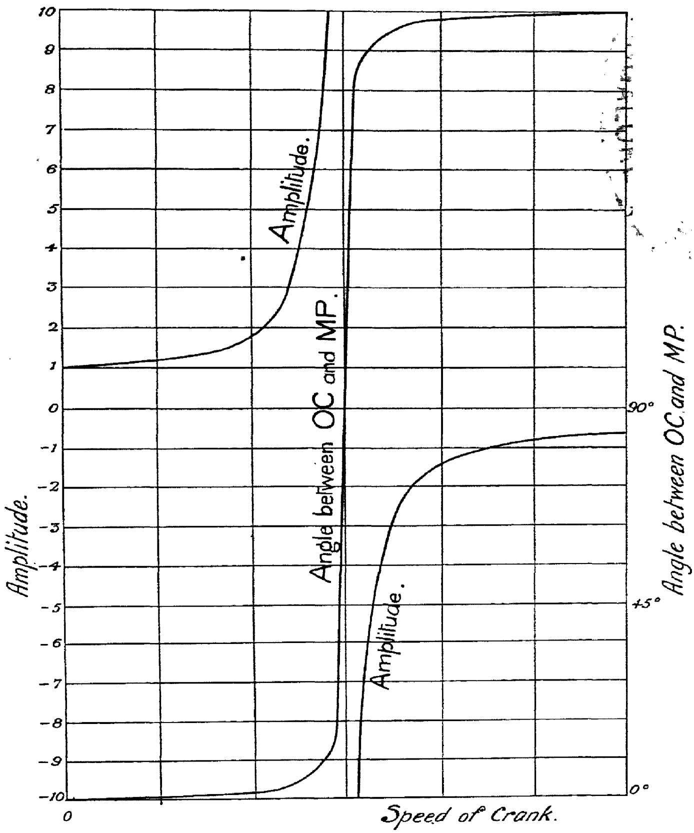


FIG. 3.

This, perhaps, can be better demonstrated by means of curves. Let us suppose that the top of the spring is attached to a crank (see Fig. 2), and that the mass M moves up and

down the diameter of an imaginary circle, upon the circumference of which its motion can be projected—the projection point on the circle moving round the circumference with uniform velocity (see p. 6). If the crank at the top is revolved very slowly indeed, its motion will be exactly the same as that of the guiding point for the motion of the mass, and the two motions, cause and effect, are seen to be in phase. Now, as the speed of revolution of the crank at the top becomes higher and higher, the angle between these lines OC and MP gradually becomes larger, until at the moment when the crank is revolved round at a number of revolutions per minute equal to the natural period of the spring and mass, this angle is 90°. As the speed of the crank is still further increased the angle rapidly increases to nearly 180°. Fig. 3 represents this, and shows that the change takes place, more or less suddenly, just at the point of synchronism between the speed of the crank and the natural period of the mass and spring. In fact, it is only the existence of friction which prevents this change being instantaneous. The greater the friction the more gradual is the change of angle.

This relation between the phase of the disturbing force and the resulting motion is most important in all that follows. We shall have cause to refer to it several times again.

Numerical Examples.

The expression:

$$\text{Time of complete vibration} \quad \frac{2\pi}{\sqrt{g}}\sqrt{\frac{\text{Moving Mass in lb.}}{\text{Restoring force in pounds weight per foot displacement}}}$$

can also be used in this form to solve many practical problems.

Mass on a Spring.—For instance, suppose we have a simple spring balance and hang a piece of iron on to it and the balance reads 2 lb. and the pointer descends 3 inches $= \cdot 25$ foot. If 2 pounds weight causes a displacement of $\cdot 25$ foot, the restoring force per foot displacement $= \frac{2}{\cdot 25} = 8$ pounds weight. If we pull it down a little more and let go, the time taken for a complete vibration up and down will be

$$\frac{2\pi}{5{\cdot}66} \times \sqrt{\frac{2}{8}} = {\cdot}56 \text{ second.}$$

When the motion is not in a straight line but is an oscillation round an axis, the fundamental expression for the time of vibration takes the form :

$$T = \frac{2\pi}{\sqrt{g}} \sqrt{\frac{\text{Moment of inertia of moving mass}}{\text{Moment of the restoring forces per radian of displacement.}}}$$

It will be seen that moving mass has been altered to moment of inertia of moving mass, or Mk^2. The mass, as before, is in pounds, k being the radius of gyration in feet (see p. 5). The restoring force is now a turning moment, the force is measured in pounds weight, as before, and its moment in feet from its point of application to the centre of the oscillation. The displacement is now in radians, measured by dividing the circumferential displacement by the radius.

Compass Needle.—As an example, take a compass needle consisting of a uniform bar of weight M lb. and of length 2 l feet and having magnetic poles of strength $+m$ and $-m$,

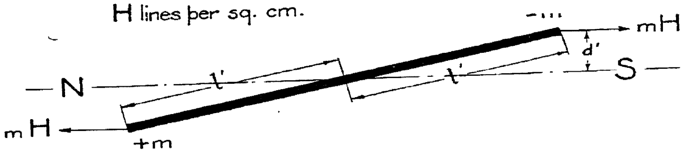


FIG. 4.

and place it in the Earth's field H to move in a horizontal plane (see Fig. 4). Each pole m in the field H will experience a force mH dynes, or

$$\frac{mH}{445{,}000} \text{ pounds weight North and South, respectively.}$$

These forces, for a small displacement of, say, d ft., will exert a restoring force having a moment

$$\frac{2mH}{445{,}000} \times d \text{ foot-lb. weight.}$$

The corresponding angular displacement in radians $= \frac{u}{l}$. So that the moment of the restoring forces per radian

$$\frac{2mHd}{445{,}000} \times \frac{l}{d} = \frac{2lmH}{445{,}000}.$$

The moment of inertia of the bar needle of mass M and length $2l = \frac{Ml^2}{3}$. So that the time of vibration

$$= \frac{2\pi}{5\cdot66} \times \sqrt{\frac{\frac{Ml^2}{3}}{\frac{2lm}{445{,}000}}} \text{ seconds,}$$

which gives either m or H, the other being known.

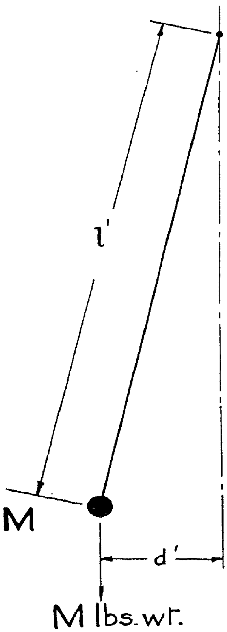


FIG. 5.

Simple Pendulum.—The time of vibration of a simple pendulum can also be very easily calculated by this method (see Fig. 5). If the bob is displaced through a small distance, d ft., the moment of the restoring force is Md foot-lb. weight. The displacement is $\frac{d}{l}$ radians, therefore the restoring moment for one radian displacement is $Md \times \frac{l}{d} = Ml$ foot-lb. weight. The moment of inertia of the bob $= Ml^2$. The time of vibration is therefore

$$\frac{2\pi}{\sqrt{g}}\sqrt{\frac{Ml^2}{Ml}} = \frac{2\pi}{5\cdot66}\sqrt{l}$$

$$= 2\pi\sqrt{\frac{l}{g}} \text{ seconds.}$$

Shaft and Flywheel.—For another example, take a mild steel shaft, 10 feet long and 2 inches in diameter, fixed rigidly at one end and carrying a plain disc flywheel, also of mild steel 10 feet

in diameter and 3 inches thick (see Fig. 6). This wheel will have a mass of

$$\pi \times 60^2 \times 3 \times \cdot 28 = 9{,}500 \text{ lb.}$$

and a moment of inertia Mk^2 of $9{,}500 \times \frac{5^2}{2} = 120{,}000$. The resistance of this shaft to twisting can be calculated, for, if C is the modulus of rigidity for the material of which the shaft

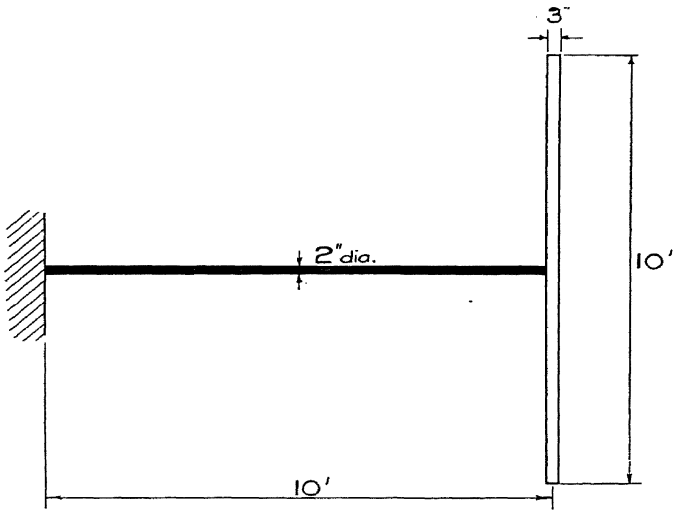


Fig. 6.

is made in lb. weight per square foot, and l the length of the shaft in feet and I the section modulus of the shaft in $(\text{ft.})^4$ units, and c the moment in foot-lb. weight to twist the shaft through one radian (supposing such a thing possible without straining the material beyond its elastic limit)

$$c = \frac{C \times I}{l}.$$

For mild steel or wrought iron C is in the neighbourhood of 17×10^8; I for the shaft section = the area of

the section in square ft. $\times \frac{1}{2}$ (square of its radius in feet) $=$ $\cdot 75 \times 10^{-4}$,

$$l = 10 \text{ feet},$$

$$c \text{ for the shaft in question} = \frac{17 \times 10^8 \times \cdot 75 \times 10^{-4}}{10}$$

$$= 13{,}000 \text{ foot-lb. weight per radian.}$$

If the shaft is already in existence this quantity c can be found by direct experiment. A convenient way of doing this would be to wind a wire round the edge of the flywheel and hang a known weight on to it, and measure the inches circumference through which the wheel revolves under its action—the other end of the shaft being, of course, fixed.

If, for example, a weight of 100 lb. was suspended, the circumference of the wheel would move through about 2·3 in. The moment of this force would be 500 foot-lb. weight, and the angle $\frac{2\cdot 3}{60}$ of a radian. The torque per radian would therefore be:

$$500 \times \frac{60}{2\cdot 3} = 13{,}000 \text{ foot-lb.}$$

The time of vibration would therefore be

$$\frac{2\pi}{5\cdot 66} \times \sqrt{\frac{120{,}000}{13{,}000}} = 3\cdot 4 \text{ seconds.}$$

More than one Flywheel on Shaft.

If an exactly similar flywheel were attached to the other end of this shaft, and the two were set to vibrate in opposite directions, then it is clear that a point of the shaft midway between the two wheels would remain stationary, and that, therefore, the restoring force per radian would be doubled, the active length of shaft being halved. The time of vibration of such an arrangement would be

$$\frac{2\pi}{5\cdot 66}\sqrt{\frac{120{,}000}{13{,}000 \times 2}} = 2\cdot 4 \text{ seconds.}$$

Such a combination of shaft and flywheel would be particularly liable to be set into vibration by an engine, and many cases

similar to this were met with in the early days of high-speed single-acting engines, where shafts, amply strong enough to transmit any load called upon, were repeatedly broken. Cases showing how the periodic time of different combinations of elasticity and inertia may be calculated are given in Part II. of this book.

Other Kinds of Shaft Vibration.

A shaft, however, can vibrate in more ways than this. It can have a transverse vibration which produces an effect like

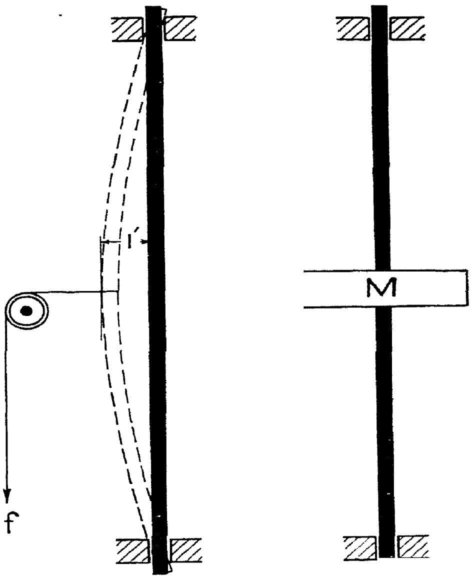


Fig. 7.

a fiddle-string, but, in the case of the shaft, the restoring force is the stiffness of the shaft, and not a stretching at the ends as with the fiddle-string.

Imagine a perfectly straight shaft supported vertically in swivelling bearings (see Fig. 7). To measure its stiffness when stationary, pull it in the middle. The displacement is found

to be proportional to the force. Let $f =$ the force in pounds weight per foot of displacement. Assume the shaft to be weightless and a perfectly balanced pulley of mass M pounds in the centre. This shaft will then vibrate like a fiddle-string with :

$$\text{Time } T = \frac{2\pi}{\sqrt{g}}\sqrt{\frac{M}{f}} \text{ or } \frac{1}{2\pi}\sqrt{\frac{fg}{M}} \text{ vibrations per second.}$$

Now rotate this shaft at n revs. $= 2\pi n = \omega$ radians per second. If subjected to no displacing force it will run perfectly at all speeds. If, however, it once gets displaced, say d feet, there will be a centrifugal force $\frac{M\omega^2 d}{g}$ pounds weight outwards. This will be opposed by the restoring force fd, which will bring it back to straightness, if fd be greater than $\frac{M\omega^2 d}{g}$, *i.e.* if ω be less than $\sqrt{\frac{fg}{M}}$, *i.e.* if n be less than $\frac{1}{2\pi}\sqrt{\frac{fg}{M}}$ revs. per second (which is seen to be the same as the natural period of the shaft in vibrations per second).

If the shaft is horizontal, the effect of gravity on M is to produce a deflection $= \frac{M}{f}$ ft.

In this case, for speeds under $\frac{1}{2\pi}\sqrt{\frac{fg}{M}}$ revs. per second the shaft will not straighten itself, but will continue to sag under the weight of M. It is important to realise that, although the shaft appears to keep the same form, it actually bends during each revolution in much the same way as the stationary shaft bends when vibrating like a fiddle-string, so that the actual motion of the revolving shaft is compounded of a simple revolution and a simple forced vibration. It is when the period of this simple forced vibration becomes equal to the natural period of the shaft that things begin to happen.

Whirling of Shafts.

When the revolutions per second are approaching the natural frequency in vibrations per second, the first disturbance takes the form of a vibration in space, that is to say, not only is the

shaft revolving and vibrating as above described, when it was apparently stationary in space, but added to these two motions there begins to be a visible vibration, often in a more or less horizontal plane. If the speed is rapidly increased, this period can be run through with safety, but if time is allowed for energy to build up at this frequency the vibration first becomes more violent and then, quite suddenly, the shaft begins to whirl, that is, to go round in space in a more or less circular path. Once this motion begins, it is, generally speaking, impossible to get through it to the safe higher speeds beyond. Once the whirling commences it will generally continue until a fracture takes place somewhere. It may be in the shaft, or, if it is a lineshaft, in the hangers supporting it.

Now suppose that the vertical shaft was not straight to begin with, but had a permanent set r ft. from the straight.

Under the action of centrifugal force, the shaft will further bend, say, d ft. The restoring force due to the stiffness of the shaft will still be fd. It is generally, but quite incorrectly, supposed that the total displacement will be $d + r$, and the centrifugal force is written

$$\frac{M\omega^2(d + r)}{g}, \text{ and this is equated to } fd,$$

from which $n = \frac{\quad}{2\pi}$ and $d = \frac{4\pi^2 M n^2 r}{gf - 4\pi^2 M n^2}$.

If, from this expression, $d + r$ is plotted against n, it is seen to be quite small (see Fig. 8), except where n approaches the critical value given by

$$4\pi^2 M n^2 = gf,$$

which is the natural frequency of transverse vibration of the shaft. At values over this, $d + r$ is again small, and of opposite sign.

This is actually very similar to what happens in practice, but in spite of all that has been written on the subject of the whirling of shafts, it is doubtful if the mechanism of the phenomenon is generally understood, and the following is put forward in the hope of clearing up points which may remain

obscure after a reading of more elaborate, but, in many respects, inadequate treatments.

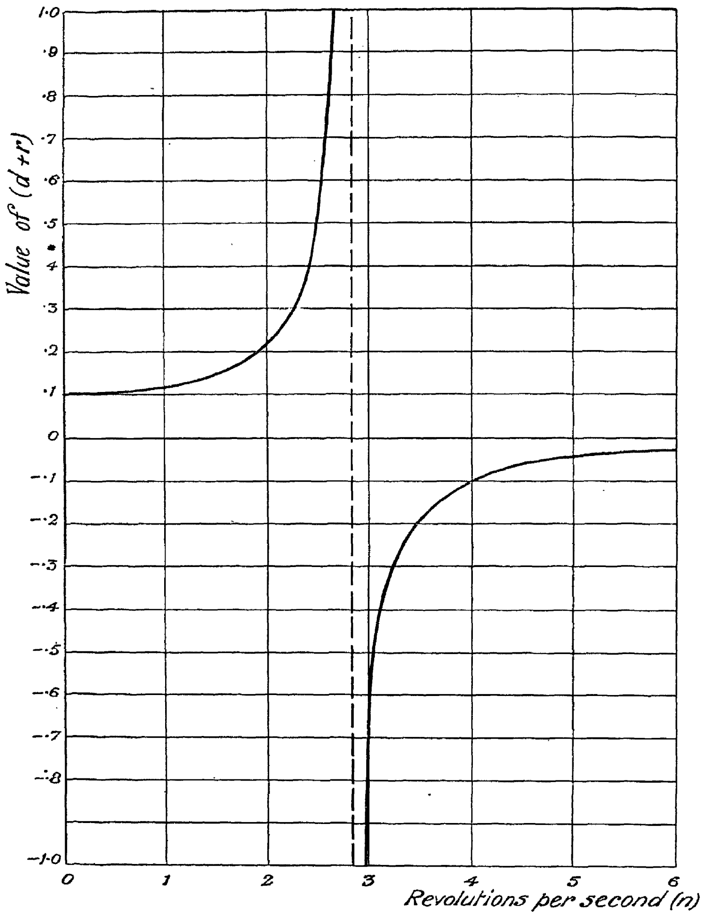


FIG. 8.

Engineers are familiar with the facts, as stated above, that, if a shaft is rotated with increasing speed, it deflects gradually

from its position of rest, but remains more or less stable until a "critical" speed is reached, at which the deflection rapidly increases. If run continuously at this speed the shaft will suffer serious damage. If, however, it is run up rapidly beyond this critical speed the deflection decreases, and the conditions are again stable. If the shaft has an initial set, or carries a mass with an initial eccentricity, it is found to run even truer at speeds above the critical than at speeds below. It is this last phenomenon that in most treatments is inadequately explained. Usually refuge is taken in some such phrase as "above the critical speed the system tends to rotate about its centre of mass," which, although true as a statement of fact, by no means constitutes an explanation. For the case of an assumed massless shaft carrying a concentrated mass, the arrangement having a slight initial lack of symmetry, the very simple expression just used and plotted on Fig. 8 is almost universally put forward to give the amount of asymmetry at all speeds both above and below the critical speeds. The expression does give the correct critical speed at which the deflection tends to infinity. It also indicates that the shaft runs truer at speeds above the critical, although it is usually derived from conditions assumed to obtain at speeds below the critical. Its application to shafts running above the critical is not, however, warranted without a clearer explanation of the physics of the phenomenon than is usually given.

It should be observed that it could equally well have been derived for the motion of a spring-controlled centrifugal governor, but we are not aware that it has been used in this connection to predict that at speeds above the critical the governor balls will move inwards instead of outwards and turn on more steam, and it is more than doubtful if such could ever be the case.

It is generally recognized that there is a resemblance between whirling and vibration, and the recognized method of obtaining the critical whirling speed is to calculate the natural frequency of transverse vibration of the system. This coincidence is not accidental, the phenomenon of whirling being essentially one of forced transverse vibration; when whirling is so regarded, the stability at speeds above the critical is very simply explained.

It will be evident that when a deflected shaft is rotated, although the material is strained by the centrifugal forces, the stresses on any portion of the shaft are constant in direction and amount. There is no alternation of stress as in an ordinary case of transverse vibration, and, at first sight, there is little resemblance. On the other hand, if two transverse simple harmonic vibrations of equal periodicity and equal amplitude act at right angles on a stationary shaft, the path of the shaft axis is a circle about its position of rest. In this case there is, of course, an alternation of stress at any given point in the material, and the positions of maximum stress or of any other constant stress move round the section with the frequency of the impressed vibrations. If, on these vibrations, there is now imposed a rotation of the shaft of the same speed as the frequency of the vibrations, there will be no stress alternation, and the conditions are exactly those of a whirling shaft.

It is, therefore, seen that the motion of a rotating shaft in "whirl" consists of: (*a*) a rotation about its axis with whatever initial eccentricity there may be; (*b*) two simple harmonic transverse vibrations of equal amplitude, acting at right angles with the same frequency as the speed of rotation of the shaft.

The total deflection of the shaft will consist of the vector sum of the initial eccentricity and the half amplitude of the vibrations.

As to some readers the identification of a whirling shaft as a case of vibration may still appear difficult, the following considerations are put forward, more to accustom the mind to this idea, than as rigid proofs, which will appear later.

Place in the drill-chuck of a lathe a piece of steel wire with a small mass at one end—a hatpin will do. Run the lathe at varying speeds till the mass runs out and whirls. Note the speed (which is the same as the natural frequency of the hatpin). Now take the pin out of the headstock, and fit into the tailstock; it will vibrate badly when the lathe is run at this same critical speed, and at no other—thus showing the vibration caused by the circular motion but separated from it.

Take, as another example, the stanchion of the building referred to later and push it backwards and forwards. Beginning slowly, the vibration produced is small because of the

dissimilarity of the two frequencies. As these get nearer, the stanchion vibrates more and more until the disturbing force equals in frequency the natural frequency of the stanchion. The resultant large motion is not only due to the frequencies being equal, but is more due to the phase relationship between them. To prove this, suddenly miss half a phase of the disturbing force, keeping the two frequencies still equal. The result will be that the stanchion will be brought to rest. As the frequency of the applied force becomes greater and greater, the two become more and more out of phase, until, at last, the disturbing force becomes the restoring force, *i.e.* the stanchion is being pushed back as it is coming away from its position of rest. This is what happens when $\frac{M\omega^2}{g}$ becomes greater than f in the example plotted in Fig. 8.

A harmonograph or twin pendulum serves to illustrate the combining of vibrations, and is very useful in studying the effect of difference of phase of two vibrations on the resultant motion. The instrument consists, in its simplest form, of two pendulums vibrating in planes at right angles; one pendulum carries paper and the other a pen. Frequencies are adjusted by alterations of length, the phase being controlled by one pendulum releasing the other at predetermined points in its swing.

It is most unlikely that the fact of the critical speed of the shaft in whirl being the same as the natural period of the shaft in transverse vibration is merely a coincidence. Any circular motion can be made up, one might almost say is made up, of two harmonic motions at right angles; as can be shown in a multitude of ways, *e.g.* by means of the harmonograph.

The actual amplitude of the whirl—first increasing as the speed increases till it reaches the shaft's natural frequency, then decreasing as the speed is further increased—follows exactly analogous laws to those governing the amplitude of a vibrational system under the action of a disturbing force of variable frequency. See p. 8 and Figs. 2 and 3.

If a shaft, which, if revolved freely, exhibits the phenomenon of whirling, is constrained by fixed parallel guides at right angles to the shaft, with their centre line passing through the

centre of revolution of the shaft, and if, under these conditions, the shaft is rotated, it will take up a rectilinear vibration, the amplitude of which follows exactly the same laws as the whirl does when the shaft is not so constrained, the frequency of the vibration being the speed of rotation.

We have, therefore, proved the existence, under the conditions described, of a periodic disturbing force, not only capable of, but actually setting the shaft into vibration. This force must have its origin in the rotation of the shaft. We can measure its maximum value and its phase relationship to the other quantities at work, notably to the displacement. We thus find all we require to explain all the phenomena which took place between these fixed guides.

Now the fixed parallel guides used in the above merely resolve the circular happenings into one of their two components, as mentioned above. Turn them round through 90 degrees and you have the other component, exactly similar, but at right angles, to the first. Take the guides away and the two combine into a circular vibration. The harmonic disturbing force becomes its radius vector, whose actual magnitude and its position relative to the whirling shafts are both known.

In case this circular motion cannot be recognized as a vibration, a very simple experiment will be convincing. Fix a hatpin by its point in a vice, or drive it into the bench; it can be set into rectilinear or circular vibration at will.

Again, by artificially preventing a whirling shaft from vibrating in its own way, the happenings characteristic of vibration at speeds above the critical value do not occur. This can be proved by a simple experiment, very similar to that described above, except that now the rectilinear guides revolve with the shaft, thus preventing the disturbing force from getting out of phase with the displacement. The shaft flies out at a speed when $\frac{M\omega^2 d}{g}$ exceeds fd, but will on no account come back to truth at any higher speeds. This illustrates the case of the centrifugal governor mentioned before, and explains why it does not behave like a whirling shaft.

To obtain the characteristic behaviour of coming in again at speeds higher than the critical, it is an essential condition that

the displacement should be able to lag 90 degrees behind the disturbing force at the critical speed, and to gradually increase this angle to nearly 180 degrees as the speed still further increases. This cannot happen if the guides revolve with the shaft.

The fact that the form of the mathematical expression for whirling does not give any indication of vibration is sometimes put forward as a difficulty in accepting whirling as a vibration, but we know of no special reason why mathematics should be obliged to explain the exact mechanism by which the results it predicts are accomplished.

Having thus considered the similarities between vibration and whirling, we come to the conclusion that the whole phenomenon of whirling may be explainable by assuming that whirling is a combination of two rectilinear vibrations at right angles, plus the rotation of the bent shaft carrying the mass at its centre, the mass being considered to be true with respect to the shaft, but being displaced a distance r from the centre of rotation, due to the shaft being bent.

The two sets of conditions will be discussed separately, namely, the rotation of the mass M in the circle of radius r, for which no elastic forces are called into play at all (this would be what would happen if the shaft were perfectly rigid). As the shaft is not rigid but elastic, there will be a further elastic displacement d.

It might be possible to consider the resultant motions together as one happening, but the expressions would be very cumbersome, and it is better to look upon it as two phenomena superimposed on each other.

If the shaft and mass were perfectly true there would be no vibration or whirling at all; there would be one speed at which, if the shaft were displaced by some outside blow, it would not return to its true position. At speeds either above or below this it would eventually run true again when the energy due to the blow had been dissipated in friction. So that it is the fact of the initial displacement r which is the primary cause of the phenomenon under consideration. Directly the shaft begins to turn there is a force $\frac{M\omega^2 r}{g}$ lb. weight, which is sometimes called centrifugal force, but which, of course, is,

rightly speaking, the force which the shaft exerts upon the mass, thereby preventing it from obeying Newton's first law, and making it travel in a circle instead of in a straight line.

We shall consider that this force exists all the time the shaft is turning, increasing with the square of the speed of rotation, and that this is the force which sets up the disturbing force that results in the elastic vibration of amplitude d.

In this vibration (as apart from the circular motion of radius r) the disturbing force $\frac{M\omega^2 r}{g}$ has to look after a lot of different jobs. It has, for instance, to overcome the stiffness of the shaft. This component is equal to fd, but is always outwards from the original configuration of the shaft, and has its maximum value when d is maximum, and is in phase with the displacement d, so that its vector can be used to point to the position of the shaft in space at any instant. The disturbing force has also to provide a component to overcome the friction of the mass M moving through the air, which force we will assume to be proportional to the velocity, or μv, where v is the velocity of M in feet per second. Its maximum value is $\mu\omega d$, which occurs when the velocity is a maximum, *i.e.* when the elastic displacement is zero (only one of the two transverse vibrations is being considered at the moment). Having overcome the stiffness of the shaft, and provided for frictional resistance, the disturbing force has also to provide the acceleration $\frac{M\omega^2 d}{g}$, which is always directed towards the centre, and is therefore seen to be 180° out of phase with the component of the disturbing force fd; both of these are at right angles to $\mu\omega d$. Therefore

$$\frac{M\omega^2 r}{g} = \sqrt{\left(fd - \frac{M\omega^2 d}{g}\right)^2 + (\mu\omega d)^2}$$

or

$$d = \frac{M\omega^2 r}{\sqrt{\left(f - \frac{M\omega^2}{g}\right)^2 + (\mu\omega)^2}}$$

The angle φ between the maximum value of the disturbing

force and the position of the shaft in space is the angle whose tangent is

$$\frac{\mu\omega}{f - M\omega^2}$$

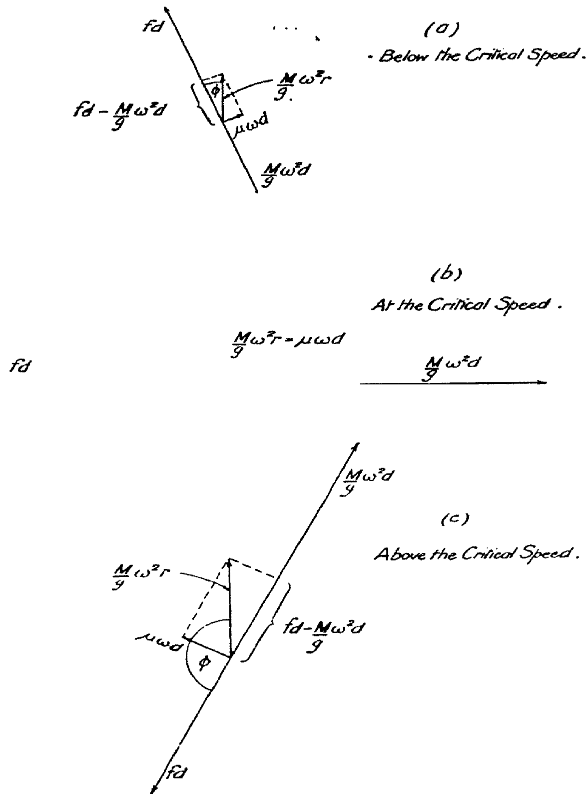


FIG. 9.

The natural frequency of transverse vibration of the shaft is

$$\frac{1}{2\pi}\sqrt{\frac{fg}{M}}.$$

Writing α for 2π times this frequency, the above expressions become

$$\frac{M\omega^2 r}{g} = d\sqrt{\left(\frac{M}{g}(\alpha^2 - \omega^2)\right)^2 + (\mu\omega)^2}$$

$$\text{and } \tan\phi = \frac{\mu\omega}{\frac{M}{g}(\alpha^2 - \omega^2)}.$$

All this applies to each of the two elastic vibrations only, which, compounded at right angles, produce that portion of the circular whirl which is due to the elasticity of the shaft. The actual motion of the shaft in whirl is this circular motion, plus that circular motion of radius r due to the shaft being bent. All the quantities discussed which, viewed in the light of a rectilinear vibration, varied harmonically as the sine or cosine of the guiding circular motion, become revolving vectors of constant length (for a particular value of the angular velocity) when the two rectilinear vibrations are combined into circular motion, $\frac{M\omega^2 r}{g}$ pointing to the original position of the mass on the bent, unstrained shaft, fd pointing to the actual position of the shaft in space. Looked at from the point of view of the elastic displacement alone, all these vectors revolve from a centre where $d = 0$ (the locus of this in the actual combined whirling motion is on the circumference of the circle of radius r). The actual application of these forces is, of course, on the mass M—the restoring force and the centrifugal force acting radially, and the frictional force acting circumferentially at right angles to the former, as was also shown in the rectilinear case.

The actual radius R of the mass M in whirl is compounded of d and r (see Fig. 10). The friction due to the combined motion in the circle of radius R is $\mu\omega R$. This has a moment $\mu\omega R \times R$ round the centre of turning. This moment is counteracted by the torsional stiffness of the shaft, which can be expressed as k foot-lb. weight torque per one degree twist of the shaft between the nearest bearing and the whirling mass M. The air friction on M will, therefore, cause the shaft to twist through an angle $= \theta$

$$= \frac{\mu\omega R^2}{k} \text{ degrees.}$$

This angle can obviously be anything, according to the

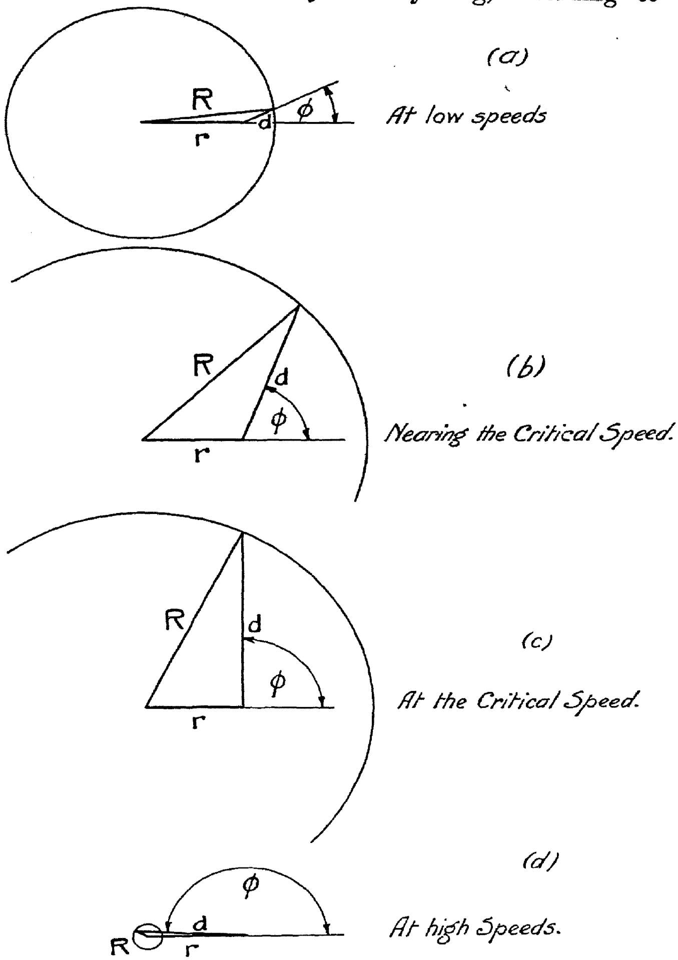


FIG. 10.

relation of the torsional stiffness of the shaft to the friction it has to overcome.

To show how this will work out in practice a numerical example has been taken and plotted on Fig. 11. For this we have assumed a mass M of 100 lb., a stiffness factor f of 310 which means that it would take a force of 310 lb. weight to deflect the centre of the shaft one foot from its position of rest. This gives a natural frequency per second of

$$\frac{1}{2\pi}\sqrt{\frac{fg}{M}} = 1 \cdot 6 \quad \alpha = 10.$$

The shaft is bent initially so that it is 0·01 foot from the straight line in the centre. The torsional stiffness of the shaft k is 31, meaning that this number of foot-lb. weight would twist it through one degree and the coefficient of friction μ of the mass M through air is 15·6, meaning that it takes a force of 15·6 lb. weight to move M at 1 foot per second.

These quantities are worked out in the following table and plotted in Fig. 11.

Fig. 12 is the same thing expressed differently, showing the various circular paths in which the mass M will move at different speeds.

ω	r.p.m.	d ft.	ϕ	R ft.	θ
1	9·5	·0001	2° — 52′	·0101	·00005°
5	47·5	·0032	18° — 26′	·013	·0004°
7	67	·0079	34° — 37′	·017	·00102°
9	86	·0166	66° — 57′	·022	·0023°
10	95	·02	90°	·022	·0025°
11	104	·0205	110° — 53′	·019	·0021°
15	143	·0155	149°	·009	·0006°
20	190	·012	161° — 34′	·044	·0002°

It is not claimed that these qualities are the property of any

actual shaft. The figures have been chosen rather with the object of showing how the different quantities react on each other, and the results would be only modified in shape and magnitude, not in principle, by the choice of any other constants.

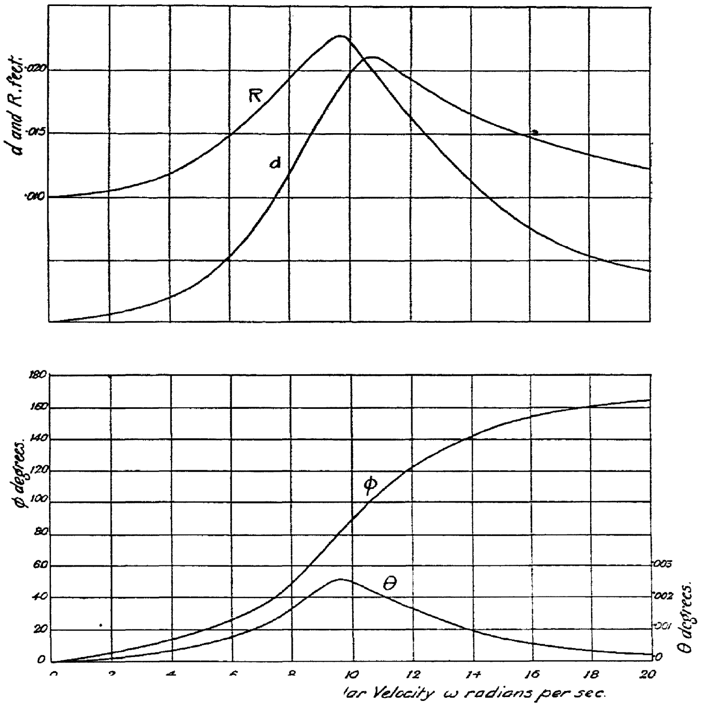


FIG. 11.

In comparing Fig. 11 with Fig. 8, the effect of friction is very noticeable. In Fig. 8 the displacement would go to infinity at the critical speed. In the present case, the displacement is finite, and in the neighbourhood of $\frac{1}{4}$ inch. The angle ϕ between the disturbing force and the displacement is seen changing gradually from about nothing to about 180°.

Without friction this change would be instantaneous. As with Fig. 8, the shaft is seen to run truer at speeds a good deal above the critical speed than it did at very low speeds.

By treating the whirling motion as a circular motion compounded on two rectilinear vibrations superimposed on a circular motion of the original eccentricity, we have explained the whole phenomenon of whirling by the comparatively simple laws of forced vibration of a system having a natural frequency, under the disturbing effect of a force having a variable fre-

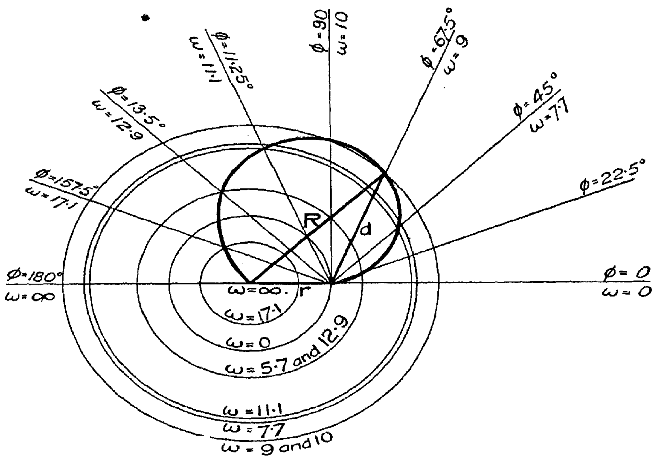


FIG. 12.

quency. It is interesting to notice the effect of friction. It is seen that the friction due to the elastic vibration is dealt with by the restoring and disturbing forces in combination, but that the friction due to the ultimate combined circular motion through space has to be borne by the source of energy which is rotating the shaft. The coefficient of friction μ, which appears in the first expressions, giving the magnitude of the elastic displacement and the phase displacement between cause and effect in the elastic vibration, also appears in the final expression giving the energy required from the external source of power to rotate the whirling shaft through space.

It is also interesting to notice that the acceleration necessary to maintain rectilinear elastic vibration is identical with our old friend $\frac{M\omega^2 r}{g}$, which in circular motion is misnamed centrifugal force. When the frequency of the disturbing force is exactly equal to the natural frequency of the system, this acceleration to the centre $\frac{M\omega^2 R}{g}$ is exactly provided by the stiffness of the shaft fR, because ω^2 then $= \frac{fg}{M}$. Under these conditions the shaft is momentarily robbed of its stiffness, and the least force at the right frequency would give it a very large deflection, only limited from being infinity by the effect of friction.

By the study of the angle ϕ, which is seen to increase from 0 for values of ω under the critical speed, and reach 180 for values over the critical speed, and to change more or less gradually at the critical speed from one of these values to the other as the friction is more or less (with no friction it would change instantaneously), it is seen that, for frequencies of the disturbing force below the natural frequency of the system, the disturbing force aids the centrifugal force in overcoming the stiffness of the shaft. Above the critical speed, by the above-mentioned action of ϕ, the disturbing force aids the restoring force to overcome the centrifugal force.

If this is compared with the action of a disturbing force producing rectilinear vibration, it will be seen to be perfectly normal, and what one would expect. It is largely the unfortunate name, "centrifugal force," given to the acceleration towards the centre when the two vibrations are combined into a circular motion, which makes it difficult to believe that a shaft can of its own accord run true again at speeds much higher than the critical.

It will now be seen why the expressions on page 17 were wrong in that it was there assumed that d and r were in the same direction and could be added arithmetically, whereas it is seen from Fig. 10 that they are only in the same straight line twice, namely, at speeds of zero and infinity, respectively, and that at the critical speed they are at right angles.

Other Ways in which a Shaft can Vibrate.

There is also possible a longitudinal vibration which could be caused by pulling a shaft with a plain stretching force and letting go, when the elasticity of the material returning to its former length combined with its mass will produce a vibration, which will generally be of a very short periodic time, as the restoring forces are very large and the moving mass is not great.

All these vibrations of a shaft can take place either with the shaft stationary, or when it is revolving. In the latter case, the vibration would be superimposed on the continuous rotation. Such a vibration could easily be set up by the uneven turning moment either of the engine driving the shaft or of some machine which the shaft was itself driving. The transverse vibration is very often started by a pulley on the shaft being out of truth. The shaft can then vibrate in as many different ways as a fiddle-string has different notes according to where it is damped, the nodes not necessarily being in the bearings. The longitudinal vibrations are more difficult to set up, but this can be done by rubbing the shaft up and down its length.

Resonance.

None of these vibrations is very harmful or dangerous to the engineer, except when the disturbing force has a frequency equal or near to the natural frequency of vibration of the system. When this occurs, however, the amplitude of the vibration is apt to build up until it becomes objectionable in many different ways, either by the fracture of the shaft, or by increasing the cyclic irregularity of the prime mover, or by shaking the building in which it is running, or by the emission of a disagreeably loud note.

Methods of Damping Vibration.

The most effective way of dealing with any class of vibrations is a method which is also applicable to certain individuals, namely, force them to do work. This is the reason why motor-car springs are built up of separate laminæ which slide one upon the other when the car vibrates, the energy of the

vibration being absorbed in doing work against the friction between the various leaves of the spring. The effectiveness of this work in absorbing the energy of vibration is proved by polishing and lubricating the leaves of the spring, when they immediately begin to transmit more of the vibration to the car body, instead of absorbing the energy in themselves. The analogous treatment, in the case of a machine, and one which is often effective, is to mount it on felt pads, the vibration being absorbed by the internal friction it sets up in the structure of the felt.

This treatment is equally effective when the vibrations are in the form of sound.

Similarly, the shock caused by power hammers is absorbed by surrounding their foundations with a trench filled with loose rubble, which has an exactly similar effect.

In the case of the mariner's compass, the vibration of the needle, which would be objectionable on board ship, is overcome by encasing the compass needle by a copper band ; the moving of the magnetic needle induces electrical eddy currents in this copper, the energy for which eddy currents has to be supplied by the energy of vibration of the needle. We shall see later that this method has a very important application in the damping out of cyclic irregularity in an alternating current generator. Another method of reducing the vibration by forcing it to perform work is exemplified by the dash pot of oil which restrains the vibration of the pointers of many measuring instruments.

Protection of Instruments from Vibration.

In the case of delicate instruments, which have to be protected from vibration, a combination of inertia and a yielding substance of large internal friction is generally the most effective. Alternate layers of paving stones and felt pads are a practical application, giving as they do almost entire immunity from the effects of vibration.

Foundations for Machinery.

Exactly similar methods to the foregoing are employed in making the foundations for machinery, especially for power

hammers and drop stamps and similar machines. Here it is often a question of a compromise between the quality of the work done and the life of the anvil or die, as the case may be. If the foundation is made too rigid, these invariably suffer. If, however, the die or anvil is allowed to yield too much to the blow, the quality of the product usually suffers.

Similarly, if the mass of the foundations is not sufficiently great, too large a proportion of the energy of the blow is dissipated, and may make itself apparent in vibration of the surroundings. To overcome this, and also to prevent the blow shattering the dies or anvil, blocks of wood are generally interspersed between these masses and the foundation. Oak, or, better still, water-grown elm is used for this. A ratio of weight of hammer to weight of block of about 1 to 15 is usually employed.

To summarize the position: a very heavy block, interleaved with wood for elasticity, and surrounded by loose rubble to prevent vibrations being transmitted to the neighbouring soil, is the ideal both for absence of vibration and for quality of work produced. If, however, the life of the anvil or dies is unduly curtailed, the mass must be reduced and replaced by more elasticity, in some cases even rubber pads being used.

The U-Tube; Calculation of its Natural Period.

In connection with instruments, the case of the U-tube for the measurement of small pressures is interesting. This has sometimes to be used in connection with pulsating pressures, when the resulting swing of the liquid in the tube becomes very troublesome. There are two methods of attacking this problem, one by inserting fairly long felt pads or wads in the connecting air tubes, thus employing the method of damping by compulsory work. It may be here mentioned that the mere closing of, say, a cock in the connecting tubes is seldom effective, as it does not give that opportunity for performing work which is so essential.

The second method is to give the liquid in the U-tube a natural period of vibration very different from that of the pulsations, so that it will be as little affected as possible by them.

To do this, it is necessary to be able to calculate the natural period, which, fortunately, is quite easy on the principles already laid down. For let m be the mass in lb. per foot of the liquid in the tube, and l the length in feet of each limb of the U-tube. The mass in motion is then $2ml$ lb., and the restoring force per foot of displacement $2m$ lb. (the liquid in one limb goes 1 foot up, and, in the other, 1 foot down). Therefore, the time of one complete swing in seconds is

$$\frac{2\pi}{\sqrt{g}}\sqrt{\frac{2ml}{2m}} = \frac{2\pi\sqrt{l}}{5\cdot 66}.$$

There are one or two points of interest in this. One is, that it is exactly the same expression as that for the time of swing of a simple pendulum (see p. 12) and another that the time of swing is independent of both the diameter of the tube and of the liquid used, *i.e.* is the same for water or mercury of the same length.

The length of the tube does not, of course, affect the readings of pressure, and, therefore, can be made such as to adjust the natural time of vibration so that the pulsations have no appreciable effect. If the length is long it need not be vertically downwards, but may be horizontal or coiled up out of the way.

Wave Form of Recurrent Force.

The effect of a recurrent disturbing force on a system the natural period of which is near to the period of the disturbing force is also affected by the precise form in which the force is applied. For example, electrical engineers will realize that alternating current forces, although of the same fundamental periodicity, may have many different wave forms, and musical people will realize that a note emitted by an organ will have a different sound when played upon a piano. The same is true with engineering disturbances. The stanchion of a building might be connected by a rigid connecting rod to the crank-pin of an engine of, say, 100 r.p.m., in which case it would receive a disturbing force of frequency 100 per minute, but the disturbance would be delivered in a perfectly rhythmic manner in conformity with the law of harmonic motion. It is easily seen

that this would produce a different effect on the structure of the building from that of 100 blows of a hammer given uniformly in each minute to the same spot on the stanchion. Each, however, would be correctly described as a disturbing force of a frequency of 100 per minute, and differences such as are shown exaggeratedly in the foregoing example must always be borne in mind. For instance, the rhythmic harmonic motion in the first example would not set into resonant vibration a stanchion having a natural periodicity of 200 per minute, whereas the 100 blows of a hammer could and would do so.

Resonance of Multiple and Sub-multiple Disturbances.

The possibility of a series of impulsive forces of one periodicity effecting resonance with a system having a natural period of greater frequency in some definite multiple must, therefore, always be borne in mind. These cases, however, are not usually so disastrous as complete synchronizing, because the vibrating system only receives energy every second or third vibration, and, therefore, will inevitably die down, to some extent, by using its energy, against friction, in between the disturbing impulses.

It should be noted again in this connection that a disturbing force of a given frequency cannot effect resonance with a system having a natural frequency of less than its own in any sub-multiple.

Effect of Cyclic Speed Variation on Alternators.

A case of vibration which is often misunderstood, owing to its being on the borderland between electrical and mechanical engineering, is that of the cyclic irregularity of an engine driving an alternating current electric generator.

Sometimes in an alternator we have all the requirements for the second class of bodies, namely, those which can vibrate in a natural period of their own. We have the mechanical inertia of the moving parts, flywheel, magnet poles, etc., and we have the restoring force of the pull of the magnetic lines of force which, in a generator, are always dragging the rotor back, and, in a synchronous motor, drag the rotor forward and supply the working torque.

Mechanical Analogies.—To vibrate, however, it is not only necessary to have mass attached to elastic cords, such as magnetic lines of force, but the other ends of these cords must be, as it were, anchored. For instance, suppose a circular brush with elastic bristles be driven round in a bored hole like the stator of an alternator, these bristles, at rest, would be more or less radial, but, on driving the brush round, the friction between the outer ends of the bristles and the stator iron would cause them to take up a more tangential position, and, also, due to the friction, the elastic bristles would tend to be stretched along their length, and, therefore, provide a restoring force to the rotor. This analogy represents fairly closely what happens when an engine-driven alternator is supplying energy to a load of the nature of a furnace or lamps. There is no tendency here to exhibit a natural period of vibration. When the rotor is slightly accelerated, the other end of the elastic cords similarly accelerates, and *vice versa.*

Parallel Running.—Now assume that the alternator could be put into parallel with a large, steady-running power station. This is analogous to attaching the outer ends of all the elastic cords, or bristles, to a framework which is revolved round at the same average number of revolutions per minute, but at an absolutely constant speed during the revolution. It will be now seen that the conditions are quite different. If the engine, at one period of the revolution, tends to accelerate the rotor above the average speed, then it is held back by the elastic cords. If, on the other hand, the engine, say, misfires (if it is a gas engine), then the even sweep of the outside of the elastic cords pulls the rotor on, and, between these limits, the rotor is free to vibrate, and has all the essentials for a natural period of vibration—lines of magnetic force being closely analogous to elastic cords.

The inner and outer ends of these elastic cords, in the case of the alternator in parallel with other steady running alternators, are revolved round by different agencies, the outer ends by energy received from the bus bars, the inner ends by energy received from the engine. Like gear wheels, they are bound to stay in step, but, also, like badly meshed gearing, there is a certain freedom for the pole to wander from its correct position,

just as a tooth may vibrate in the space between the two opposing teeth.

Moreover, the system may be set into vibration, either by a periodic irregularity in the turning effort of its own engine, or by the same fault in the turning of the outer ends of the lines of force derived from the bus bars.

This vibration is purely mechanical, and has, of course, nothing to do with the frequency of the alternating current supply, or with any frequency which may exist in the electric circuits, due to their resistance, self induction or capacity. The restoring force of the magnetic pole works in just the same way as that already described for the compass needle, and, after all, the magnetic pull is no more, perhaps rather less, magical than the pull of gravity on the bob of a pendulum, so that the engineer may have no hesitation in tackling the problem. There are, however, one or two data which he will have to get from his electrical *confrère*. To solve the expression giving the natural period of vibration of such an alternator, it is necessary to know, not only the amount of the restoring force, but also the restoring force per radian of displacement. The amount of the restoring force is easily arrived at. For it is exactly equal and opposite to the driving torque exerted by the engine. To get the radians of displacement, the electrical designer will have to be asked a question, and the question may be put in this form: What fraction of a revolution are the rotor poles ahead of the stator poles for some given load in horse power? The electrical designer who knows his business should be able to reply to this question without any trouble. He will probably, however, from force of habit, give the answer in electrical degrees instead of in degrees of circumference, an electrical degree being $\frac{1}{360}$ of the angle between two adjacent north poles of the alternator magnets. In case the reader is not willing to leave it at that, the following method is given for calculating this angle.

Calculation of Natural Period of Alternator.—In a two-pole alternator let the line OV (Fig. 13) represent by its length the maximum positive value both of the terminal volts and of the number of lines of force in the stator producing them. The scales of each can be so chosen that the same length represents

both quantities, and, as the position of the maximum values of both in space is identical, OV can also represent their position at one particular instant, the line OV being supposed to rotate counter-clockwise round O as centre. In the same way, let the line OC represent the maximum positive value of the current. This lags behind OV by an angle ϕ, the cosine of which is the power factor of the external circuit being fed by the alternator.

By reason of the local magnetizing effect of the current OC, lines of force surround the stator conductors, and cross the air space 90 electrical degrees from where OC is a maximum: let OL represent this leakage flux. The flux which the rotor magnets send across the air space has to neutralize this leakage flux, and also to supply the main stator flux OV; therefore it is represented by the resultant of OL reversed, and OV or OF in the figure.

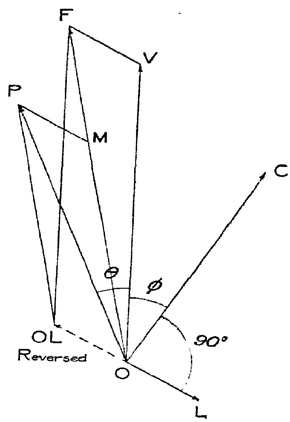


Fig. 13.

The current OC, besides producing the leakage flux OL, has a direct de-magnetizing effect on the rotor magnets, the maximum effect of which is along the radial line half-way between the maximum positive and negative values of OC, or 90° from each, *i.e.* in the same direction as OL.

As before, by a suitable choice of scale let the line OL represent this de-magnetizing effect of OC. Using this same scale, mark off on OF a length OM to represent the magnetization required for the flux OF. The rotor poles not only have to supply this, but have to overcome the de-magnetizing effect of the stator current represented by OL, and they are, therefore, represented in magnitude and direction by the resultant of OL reversed and OM shown as OP in the figure.

The magnitude of OP or the strength of the rotor pole is interesting to the designer, but it is its direction which interests us at the moment, as it will be seen that the angle θ between

OV and OP is the lag of the stator pole OV behind the rotor pole OP, for an electrical output of OV volts and OC amperes at a power factor = cos ϕ, which is what was required to calculate the restoring force per radian. The angle θ is in electrical degrees.

This natural period will obviously be altered by alterations in the strength of the magnetic field in the air space between the stator and rotor. As this strength of field is that which produces the voltage of the machine, and as, in general, alternating current generators are used for producing a constant voltage, this does not affect matters.

Effect of Power Factor.—The magnetization of the machine will have to be slightly altered for constant terminal voltage according to the power factor of the load to which the machine is supplying energy. These alterations, however, are of quite small magnitude, and are immaterial to the engineer, as the calculation of these natural periods is generally undertaken in order that any other disturbing force may be kept very well away from them. If you wish to keep a mile away from a tiger its exact position need not be known to more than a few feet. It will be seen from the expression for the time of oscillation that the quantities about which there may be some doubt are all under the square root, so that a 10 per cent. error in them will only cause a 5 per cent. error in the result. This is a fortunate occurrence, as in many of the problems met with in actual experience it is impossible to calculate either the moment of inertia of the moving mases, or the restoring forces, to any very high degree of accuracy.

As explained above, the alternator can be set into a state of vibration, either by cyclic irregularities in the turning effort of its own engine, or in the engine driving any other alternator with which it is in parallel.

In choosing an engine-driven alternator to work in parallel with existing sets, care must, therefore, be taken that the periodic time of the new alternator not only does not synchronize with any disturbing forces of the new engine, but also with any of the disturbing forces of any of the existing engines with which it will have to work in parallel, and also that none of the possible disturbing forces in the drive of the

new engine will synchronize with the natural periods of any of the existing alternators.

In multi-cylinder gas engines the number of impulses per minute to be feared is not only the total number of explosions per minute, but also the explosions per minute on any one particular cylinder, which may, either at a period, say, of light load be doing most of the work, or may, on the other hand, be consistently misfiring.

Vibrations may also be set up by any rhythmically recurring disturbance in the machinery which is being driven by any sort of electric motor. For instance, an alternator supplying energy to a motor driving an air compressor or a stamp would experience a series of retarding efforts as each momentary peak load came on.

Effects of Resonance.—If any of these retarding efforts, or the accelerating efforts due, say, to the explosions in its own gas engine, happen to be of the same frequency as its own natural period of vibration, these disturbing forces will be likely to set up a torsional oscillation superimposed on its regular revolution, which may tend to increase to such an extent as to throw it out of step with the other generators running in parallel. Even if things do not get so bad as this, the presence of this oscillation may be objectionable from a variety of causes. It may interfere with the governing of the engine. It will make the readings of all the electrical instruments unsteady, and it will at best cause the electrical supply to be of varying periodicity. For, in general, the swinging of the alternator will be much slower than the periodicity of the electric supply, so that 10 or 50 complete electrical cycles may take place during one swing of the alternator.

It would be well here to draw attention again to the fact that a single engine-driven alternator with no synchronous machinery on its mains is incapable of having a natural period of vibration, for the reason that both ends of its lines of force are under its own control, and, therefore, would both move if one moved, which state of affairs would be analogous to that of the shaft and flywheel in a previous example with the other end of the shaft unfixed.

Relation between Speed and Position Error of Engines.

As the cyclic irregularity of engines is the chief cause of this trouble, it will be well to investigate this matter more fully. Most engine-makers give the cyclic irregularity of their engines in terms of variation in speed during a revolution. It is, however, the error in position of a magnet pole from where it should be at any given instant on the assumption of uniform turning which is important. The connection between this error in speed and error in position is not always quite apparent. The problem is complicated by the length and duration of each power stroke in each cylinder of the engine by the obliquity of the connecting rod, and by the precise shape of the curve representing the speed variation between the limits given. In the following, the simplest case is first considered, namely, on the assumption that all the impulses of the engine are alike, evenly spaced round the revolution, and that the speed varies from its maximum to its minimum on a straight line or saw-toothed curve. The conditions which modify these results are studied in the second part of the book, but the simple treatment here formulated gives results which are found to be correct within 10 or 15 per cent. in actual practice.

First consider a single-cylinder, single-acting steam-engine admitting full-pressure steam for the full stroke and then opening to atmosphere. During this stroke the flywheel is being accelerated from its minimum to its maximum speed; during the return stroke the flywheel is taking the load, giving up energy by lowering its speed. So that twice in a revolution, in the middle of each stroke, the engine is running at its mean speed. Now imagine another flywheel running alongside this one, but at a perfectly uniform speed, equal to the mean speed of the engine under consideration. Suppose a chalk mark ruled parallel to the shaft across the face of the two wheels at the middle of the power stroke, *i.e.* at an instant when the speed is equal to the mean speed; for the next half-revolution until the middle of the return stroke the speed is more or less above the mean speed, but always above it, *i.e.* the engine flywheel is moving faster than the perfect one, and, therefore, the two chalk marks will be separating all this time. After this

has gone on for half a revolution, the engine speed begins to fall below the mean, and, as its chalk mark is in front of the other, the two will begin to approach each other. From this it is evident that the maximum distance apart of the chalk marks is the maximum error in position, and occurs at the middle of the return stroke. At the middle of the next power stroke the two chalk marks are together again, and the process repeats.

If R_0 be the mean speed in revolutions per minute, R_1 the lowest and R_2 the highest momentary speeds, the time during which the lines have been separating is half the time of a revolution, that is, $\frac{1}{2R_0}$ minutes. The average speed during that time has been $\frac{R_2 + R_0}{2}$ r.p.m. The speed of the ideal flywheel has been R_0, the difference in speed which is the relative speed of the two chalk marks has been $\frac{R_2 + R_0}{2} - R_0$ r.p.m. This error in speed has continued for $\frac{1}{2R_0}$ minutes, therefore the fraction of a revolution separating the two marks at their greatest distance apart is equal to:

$$\left\{\frac{R_2 + R_0}{2} - R_0\right\}\frac{1}{2R_0} = \frac{R_2 - R_0}{4R_0}.$$

It will be noticed in the above that, although the speed of the engine is alternately above and below that of the ideal wheel, the chalk mark on the engine's wheel is always in front of the other, so that the maximum error in position is really plus or minus half the above amount, that is to say, $\frac{R_2 - R_0}{8R_0}$. Now $R_2 - R_0 = \frac{1}{2}(R_2 - R_1)$. Therefore the maximum error is $\frac{R_2 - R_1}{16R_0}$ of a revolution, or, expressed in degrees of circumference, its value is:

$$\frac{R_2 - R_1}{16R_0} \times 360 = 22{\cdot}5\,\frac{R_2 - R_1}{R_0},$$

or, expressed in electrical degrees, it is:

$$11{\cdot}25 \times \text{Number of poles} \times \frac{R_2 - R_1}{R_0}.$$

$\frac{R_2 - R_1}{R_0}$ is the form in which most engine-makers give the cyclic variation of speed, which is often of the order of $\frac{1}{500}$.

For other engines, the principal information required is the number of times in a revolution that the speed is put right, *i.e.* for how long the error in speed is allowed to continue. This depends on the number of power impulses the flywheel receives per revolution. Calling this N, the error in position in electrical degrees would be:

$$\frac{11{\cdot}25 \times \text{Poles}}{N} \times \frac{R_2 - R_1}{R_0}$$

N for the engine considered $= 1$. If the single cylinder engine were double-acting it would be 2, and is still 2 for any double-acting tandem engine. For a side-by-side compound with cranks at 90° it is 4; with cranks at 180° it is 2 again; for a triple side by side engine with cranks at 120°, $N = 6$, and so on. For gas engines, a single-cylinder Otto-cycle single-acting engine would make $N = \frac{1}{2}$, and a tandem ditto 1, and so on.

These amounts of error in speed and error in position only apply, of course, to engines driving some load which does not, in itself, constitute a disturbance to the regularity of motion. When an engine is driving an alternator, which has a natural period corresponding to any of the various periods of the engine, the cyclic irregularity is, of course, increased, and, if the correspondence between the two frequencies is too close, and if no artificial method of damping the vibrations is used, the effect, being cumulative, increases as it stores up energy at that frequency until something happens. This may be the breaking of a shaft, the shearing of a key, or the wrecking of an engine, or merely the falling out of step of an alternator.

Numerical Examples.—For an example of the foregoing, a 200 K.W. 50-cycle 200 r.p.m. alternator had an angle of lag

between its stator and rotor poles at full load of 15 electrical degrees. The combined flywheel effect of its own rotor and the engine flywheel gave a moment of inertia Mk^2 of about 100,000 lb. ft.^{2} units. The alternator had 30 poles and required about 300 H.P. Three hundred H.P. at 200 r.p.m. involves a driving torque of 7,900 foot-lb. weight. Fifteen electrical degrees and 15 pairs of poles equals one degree of circumference. One radian equals 57 degrees, so that the restoring force per radian would be 450,000 foot-lb. The natural period of vibration would, therefore, be :

$$\frac{2\pi}{5{\cdot}66} \times \sqrt{\frac{100{,}000}{450{,}000}}$$

or 0·52 second, which is 116 complete swings per minute.

If a steam engine were to drive the alternator, the lowest frequency disturbance would be at 200 per minute, which would be quite satisfactory, for a little consideration will show that, though a disturbance of the lower frequency may set up a swing in a system having a higher frequency, which is an exact multiple of the other, the converse does not hold. For, imagine a pendulum swinging 100 complete swings per minute acted on, say, by puffs of air at 200 per minute; if the first puff was in the direction of motion, and so aided the swing, the second would meet the pendulum coming back, and tend to stop its motion by the same amount.

If, however, the engine were an Otto-cycle gas engine it would have a disturbing effect of 100 impulses per minute, which would be far too near the 116 to be safe, in which case something would have to be done.

Cures for Trouble.—The most obvious thing would be to alter the mass of the flywheel; here one has the choice of either increasing the inertia, and so lowering the natural frequency, or reducing the flywheel effect, and so bringing the frequency between the two danger zones round 100 and 200 per minute. Of the two alternatives the former commends itself. The latter method is too suggestive of steering between Scylla and Charybdis to be altogether comfortable, although the risk is not prohibitive should there be no other way out.

The problem can also be attacked from the electrical end

by means of an alteration in the angle of lag between stator and rotor poles. The readiest way to do this in an existing alternator is to alter the magnetizing force. If nothing else is done the terminal voltage will, of course, be affected, which may be permissible for purposes of experiment, but which can be corrected by a corresponding alteration in the magnetic circuit, either in the section of the iron in poles or yoke, or more easily in the air space, which may be shortened by packing up the poles, or altered in area by means of the pole tips. The precise way in which this alteration in magnetizing force affects the angle θ is readily seen from Fig. 13. It will be observed, in this connection, that the very circumstance we were congratulating ourselves over works both ways, in that to effect a given percentage alteration in frequency, the inertia or the angle θ must be altered twice this amount.

Damping of Alternators.—Besides these methods of running away from the danger, we can, of course, stay there and take steps to mitigate the results. Exactly the same method described as being employed to damp the vibrations of the compass needle can be used in this case. If a copper sheath or cage of copper bars be carried round by the rotor of the alternator, such of the lines of force as are behaving themselves, and going round at the same speed without any superimposed swing, will have no effect in making eddy currents in this copper, but if one end of the lines of force is, at any instance or point of the revolution, revolving at a momentarily different speed from their other ends, then this revolving copper will be cut by such lines of force, with the result that electrical eddy currents will be set up, the energy for which will have to be supplied from the energy represented by the swing of the alternator, as distinguished from the uniform revolutional speed. Such damping copper is now generally introduced into the construction of an alternator which may have to run in parallel with other alternators, or what amounts to much the same thing, to supply energy to synchronous motors. Without carrying a very excessive amount of copper, a torque equal to the full-load torque of the engine can be applied to the irregular part of the turning effort. It will, of course, be seen that these dampers only absorb energy in proportion to

the deviation of the actual speed from the ideal of uniform speed, and that, if the speed is uniform, they entail no loss of energy at all.

Other Troubles with Cyclic Irregularity.

But it is not only when engines are driving electric generators that the question of cyclic irregularity matters. Many classes of machinery are very susceptible to variations in the speed at which they are driven, notably paper-making machinery and cotton-spinning frames.

Some very accurate instruments are available for indicating the cyclic irregularity in turning, not only of the engines, but also of the far end of the mill shafting which they drive. It is often found that this irregularity becomes worse the farther one gets from the main engine drive. This may be due either to the natural spring in the line shaft and the flap or give of the belt or rope drives, or it may be due to some periodic disturbance of the engine being picked up by the natural period of some part of the line shafting. The trouble thus caused is often felt, not only in the quality of the product manufactured, but in vibrations being transmitted through the hangers to the whole structure of the building.

In some cases this synchronism of the natural period of a shaft with the beats of an engine has resulted in the fracture of an otherwise perfectly good shaft, which was amply strong enough to transmit the maximum horse power it could ever be called upon to carry; indeed, such fractures have sometimes occurred on quite light load.

A peculiarity of such cases is that the shaft can be strengthened by either increasing or decreasing its diameter. In either case, the effect is to alter the natural period, so as to remove it from the neighbourhood of the period of the disturbing force.

It is, however, in general, better practice to have the natural period of the shaft above the frequency of any disturbing force caused by the engine, so that, on starting up from rest, the system does not go through a critical speed. Many plants, however, are run with such critical speeds below their normal running speed. Serious trouble is avoided by running through such speeds quickly; the effect, being a cumulative one,

requires a certain time for the energy to build up to dangerous proportions. In many other cases, the natural period of the system can be altered by suitably placed weighted pulleys.

Experimental Measurement of Moments of Inertia.

In many of these calculations we are met with the difficulty of finding the moment of inertia of the revolving masses, such as flywheels, pulleys, armatures of electric generators, etc. Many of these are very difficult to weigh and measure, and small inaccuracies of measurement mount up very rapidly when the third and fourth powers of them are introduced into calculations.

If the actual thing can be got at and experimented with separately, there are one or two ways in which the moment of inertia may be measured. For instance, the armature of an electric generator on its own shaft may be supported on level rails, so that it is free to roll. A known mass may be attached to it at a definite distance from the centre line of the shaft. It can then be started to oscillate like a pendulum, and the periodic time counted for, say, 10 or 100 complete swings.

$$\text{T, the time of vibration} = \frac{2\pi}{\sqrt{g}} \times \sqrt{\frac{\text{moment of inertia}}{mr}}$$

where m is the added mass and r is its distance from the centre of the shaft, and T is the time of one complete swing in seconds.

In some cases it is easier to hang the wheel, or armature, by a two-wire suspension, so that the axis in which it usually revolves is vertical. The moment of inertia of the suspended masses can then be calculated from the time of torsional vibration, for

$$\mathrm{T} = \frac{2\pi}{\sqrt{g}} \times \sqrt{\frac{\text{moment of inertia} \times 4l}{\mathrm{M}d^2}},$$

where l is the length of the two wires forming the suspension, d their distance apart, both in feet, M the mass of the armature, wheel, etc., in pounds, and T the time of one complete swing in seconds.

Diagnosis and Cure for Vibration Troubles.

In investigating cases of trouble caused by vibration, unless the offending cause is quite obvious, there may be some difficulty in bringing the responsibility home to the real culprit. In this case, a great help in the preliminary investigation is to be able to measure the frequency of the disturbing vibration. For this purpose a very simple apparatus can be used.

Frequency Meter.—This instrument consists of a steel wire, with a mass of metal at one end and arranged in a clamp, so that the length of wire between the mass and the clamp can be varied (see Fig. 14). In one position of the clamp there is one particular frequency at which it can vibrate. If subjected to even a very slight vibration at the same frequency, the motion will be picked up and very largely magnified, but unless the disturbing frequency is within a few per cent. each side of the

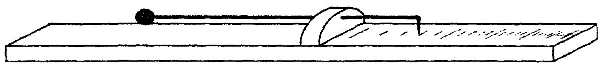


FIG. 14.

natural periodicity of the instrument as set, the disturbing force is unable to cause any noticeable vibration. The method of using this instrument is to put it down on the wall, or floor, or structure, which is suffering from the vibration, and to slide the wire in and out of the clamp until the vibration is picked up and amplified to the maximum extent. The frequency in seconds per vibration, or vibrations per minute, can then be read off from the position of the other end of the wire on a suitable scale. The instrument can easily be calibrated by placing a variable-speed motor on a table, the motor being equipped with a speed indicator, and some out-of-balance weight on its shaft. By the aid of this instrument it will then be easy to locate the cause of vibration by finding which of the possible causes has a similar frequency.

The Vibragraph.—The frequency meter, however, will not give definite indications as to the amount or amplitude of the vibration, but an instrument has been invented by Mr. Pollard Digby, and called by him the Vibragraph, which consists

essentially of a mirror floating on mercury centrally pivoted, and free to respond to any vibration in a horizontal plane. A beam of light from any suitable source is thrown on to the mirror, and from it reflected on to a ground-glass screen or photographic plate. From the size and shape of the path traced out by the spot of light a great deal may be learnt both as to the amplitude and quality of the vibration. By means of these two instruments many cases of vibration can be investigated, their source traced and a suitable remedy devised.

If the troublesome vibration be accompanied by a noise, as mentioned previously, there is the possibility of photographing the sound wave emitted, which photograph may be of much use for detecting the cause, and hence finding the remedy.

In investigating causes of trouble from vibration, the possibility of resonance should always be borne in mind, as, in very many cases, it is only when resonance occurs that vibration really becomes harmful. A particular example of this actually met with in practice was the vibration of a bunch of artificial flowers in a room, calling attention to and emphasizing vibration which was hardly noticeable otherwise. When such cases are found there is often the choice of remedying them by change of either the periodicity of the disturbing forces, or the natural period of the object set into vibration. An additional block of concrete judiciously placed against an engine bed may prove all that is needed.

Professor Perry cites an instance in which the moving of one heavy article of furniture from one side of the room to the other had the desired effect of sufficiently altering the natural period to avoid resonance. In structural work, the loading with concrete of the space between two joists has been found to do all that was necessary.

Water as a Carrier of Vibrations.—Water has been found to be particularly apt to carry vibrations, as, say, in a water-logged sub-soil, in which case either general drainage or drainage on a definite line enveloping the area affected is the best remedy, where the remedy has to take the form of mitigating the symptoms rather than of curing the cause.

Effects of Vibration on Individuals.

A good deal of work has been done on the susceptibility of different individuals to the bad effects caused by vibration. A great divergence has been found in different individuals in this quality, but it should be possible to set down the range of frequency over which vibration is particularly harmful or disagreeable to human beings, and the limiting amplitude, defined in terms of frequency, beyond which excessive fatigue or annoyance is caused.

Mr. Pollard Digby and Captain Sankey read a paper at a meeting of the British Association on "Human Susceptibility to Vibration," and recorded that they had found great differences in various individuals, but, in general, the deleterious effect could be measured by a unit, composed of the frequency multiplied by the amplitude of the vibration.

PART II. MATHEMATICAL

(See p. xi for the List of Symbols used in this part of the book.)

Definition of Terms and Units.

A linear displacement is stated in feet, the symbol being, say, x.

The unit of time is the second.

Linear velocity is stated in feet per second, and is denoted by $\dot{x}$ or v. This may be uniform or variable, and, when variable, the symbol will denote the instantaneous value; in this case, it represents the distance in feet which would be moved through if the velocity remained constant for a second.

Linear acceleration is the rate of increase of linear velocity per second, and is stated in feet per second per second. It is denoted by $\ddot{x}$ or $\dot{v}$.

An angular displacement will be stated in radians, the symbol being, say, θ.

The radian is the unit angle in circular measure, and is the angle of which the arc is equal to the radius. A complete revolution of 360 degrees has an arc $= 2\pi r$, and is, therefore equal to 2π radians. Therefore, one radian $= \frac{180}{\pi}$ degrees.

Angular velocity is stated in radians per second, and is denoted by $\dot{\theta}$, or such a symbol as ω or α.

Angular acceleration is the rate of increase of angular velocity per second, and is stated in radians per second per second. It is denoted by $\ddot{\theta}$, $\dot{\omega}$ or $\dot{\alpha}$.

As in the case of linear velocity all the above symbols may refer to a uniform condition or to an instant in a varying condition.

Relation between v and ω. Assume a particle at radius r to

move round an axis with angular velocity ω. Its tangential velocity, v, will be equal to $r\omega$.

Hence $v = r\omega$.

Relation between rate of Revolution and Angular Velocity. The rate of revolution of a shaft is usually stated in revolutions per minute. In one revolution it moves through 2π radians. Therefore, if its speed is N revolutions per minute the angular velocity in radians per second is given by

$$\omega = \frac{2\pi N}{60}$$

This quantity may also be termed *circular frequency.*

The Unit of mass is the pound, and is denoted by the symbols M or m.

Linear momentum is the product of mass and linear velocity, and equals Mv or $M\dot{x}$.

Force is the rate of change of linear momentum, and is, therefore, proportional to the product of mass and acceleration. It is denoted by such symbols as F and P. The units of mass, displacement and time having been chosen, the rational unit of force will be one for which the simple relation $F = M\ddot{x}$ holds without the introduction of a numerical coefficient. It is, therefore, that force which, acting on a mass of 1 lb., produces an acceleration of 1 ft. per second per second. This rational or absolute unit in the British system of units is called the *poundal.*

Any other unit of force may be used, provided that it consists of a definite number of poundals. One such unit of force, and that in most common use, is the weight of a pound, which is the pull exerted by gravity on a mass of 1 lb. This pull we know to produce an acceleration of g, or 32·2 ft. per second per second, as against the 1 ft. per second per second of the poundal. The pound weight is, therefore, 32·2 times as large as the poundal. Although this figure has the same numerical value as the acceleration due to gravity, it should not be regarded as such in what follows. It is rather a numerical coefficient introduced for converting the arbitrarily chosen unit of the pound weight to the absolute unit of the poundal.

Moment of Inertia.—This term is often used to represent both a geometric property of a section, and the inertia factor in problems of rotation. We propose to use it only in the latter sense, using the term *Second Moment of Area,* or simply *Second Moment,* for the geometric property of the section.

Second Moment of Area.

In problems involving bending or torsion it is necessary to know the Second Moment of Area of a section of the elastic material, with respect to a certain axis. It is denoted by the symbol I. The area of the section in Fig. **15** is A, and may be considered to consist of a number of small areas a_1, a_2, a_3, etc.

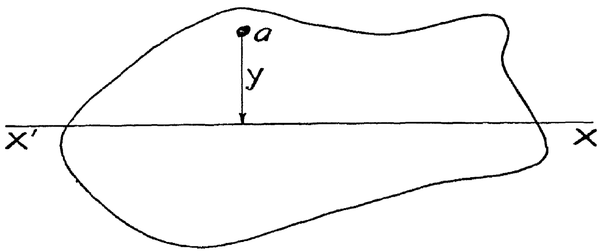


Fig. 15.

It is required to define the Second Moment of the section with regard to the axis XX′.

The second moment of each element is obtained by multiplying its area by the square of its distance from the axis, viz. ay^2, and the Second Moment for the whole area is the sum of all such quantities.

$$\mathrm{I} = \Sigma ay^2 = \mathrm{A}\Sigma y^2.$$

This quantity may easily be calculated for simple geometric forms and is expressed either in $(\text{feet})^4$ or $(\text{inches})^4$.

Moment of Inertia.

This quantity is similar in form to the second moment of area, with the difference that mass takes the place of area. It is the inertia factor in rotational dynamics.

If m_1, m_2, m_3, etc., are the masses of constituent elements in

a total mass M, and r_1, r_2, r_3 are their respective distances from the axis of rotation, then

$$\text{Moment of Inertia} = \Sigma mr^2.$$

If the whole of the mass M is assumed to act at a radius k such that

$$Mk^2 = \Sigma mr^2$$

k is the *Radius of Gyration* about that particular axis.

If the axis passes through the mass centre of the body the radius of gyration may be written $\bar{k}$ and the Moment of Inertia $M\bar{k}^2$.

The radius of gyration is expressed in feet and the moment of inertia in pounds (feet)^2.

Moment of Inertia and Second Moment of Area about Parallel Axes.

If $M\bar{k}^2$ is the moment of inertia of a body about an axis through its mass centre, it is required to find its moment of

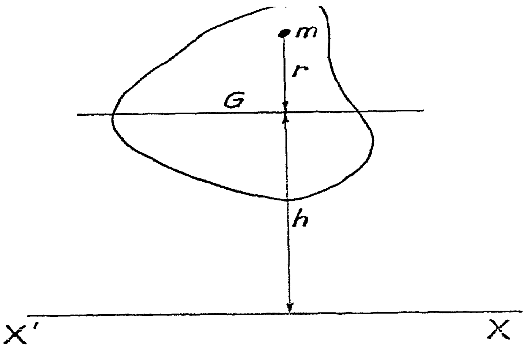


FIG. 16.

inertia, Mk^2, about any other axis XX′ parallel to it but distant h from it.

$$Mk^2_{XX'} = \Sigma m(r + h)^2$$
$$= \Sigma mr^2 + 2h\Sigma mr + \Sigma mh^2.$$

But $\qquad \Sigma mr = 0$

Therefore $\qquad Mk^2_{XX'} = \Sigma mr^2 + \Sigma mh^2$
$$= M\bar{k}^2 + Mh^2.$$

Similarly, for the Second Moment of Area we have the corresponding expression:

$$Ak^2_{XX'} = A\bar{k}^2 + Ah^2$$

i.e.

$$I_{XX'} = \bar{I} + Ah^2.$$

Second Moment of Area about an Axis normal to its Plane.

If the Second Moment of Area is known about each of two axes XX′ and YY′ at right angles, the second moment about an axis at right angles to XX′ and YY′ and passing through

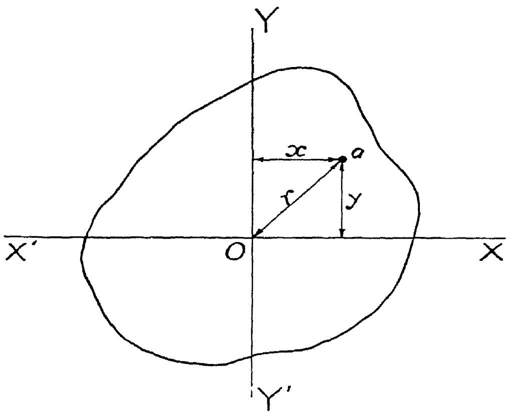


Fig. 17.

their intersection O, is equal to the sum of the second moments of area about XX′ and YY′.

For, as before, consider the area to consist of elements a_1, a_2, a_3, etc., the second moment of one such area about XX′ is ay^2, and the second moment about YY′ is ax^2. The second moment about O is $ar^2 = ay^2 + ax^2$.

And for the whole area the Second Moment about O is

$$\Sigma ar^2 = \Sigma ay^2 + \Sigma ax^2$$

i.e.

$$I_O = I_{XX'} + I_{YY'}.$$

This property is of particular use in finding Second Moments of Area of circular shafts for torsional and bending problems.

To find the Second Moment of Area of a shaft section about a central axis normal to the section. This is usually known as the Polar Moment to distinguish it from the Second Moment about a diameter.

Consider an annular element of radius r and thickness dr.

Area $= 2\pi r dr$ which acts at a radius r.

Therefore, Second Moment of element $= 2\pi r^3 dr$.

Second Moment for whole area $= 2\pi \int_{r=0}^{r=R} r^3 dr = \frac{\pi R^4}{2}$.

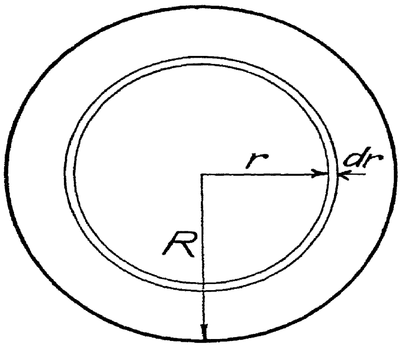


FIG. 18.

In the event of the section being hollow, an inner radius R_1, and an outer radius R_2

$$\text{Second Moment} = \frac{\pi}{2}(R_2^4 - R_1^4).$$

From these results the Second Moment about a diameter may easily be deduced from the relation

$$I_O = I_{XX'} + I_{YY'},$$

where XX′ and YY′ are intersecting axes mutually at right angles.

In the present case $I_{XX'}$ and $I_{YY'} = I$, the Second Moment about a diameter,

i. e. $$I_O = 2I, \text{ therefore } I = \frac{I_O}{2}.$$

Therefore, for the solid circular shaft,

$$\text{I, about a diameter, } = \frac{\pi R^4}{4},$$

and for the hollow shaft

$$\text{I, about a diameter, } = \frac{\pi}{4}(R_2^4 - R_1^4).$$

Linear and Rotational Analogues

Symbols and Units

Linear Motion	Rotational Motion
Displacement. x ft.	Angular Displacement. θ radians.
Velocity. $\dot{x}$ or v ft. per sec.	*Angular Velocity.* $\dot{\theta}$ or ω radians per sec.
Acceleration. x or $\dot{v}$ ft. per sec.^{2}	Angular acceleration. $\ddot{\theta}$ or $\dot{\omega}$ radians per sec.^{2}.
Mass. M or m lbs.	Moment of Inertia. Mk^2 lb. ft.^{2}
Force. F poundals. $= M\ddot{x}$.	Torque. T foot poundals. $= Mk^2\ddot{\theta}$.
Momentum. $M\dot{x}$.	Moment of Momentum. $Mk^2\dot{\theta}$.
Kinetic Energy of Translation. $\frac{1}{2}Mv^2$.	Kinetic Energy of Rotation. $\frac{1}{2} . Mk^2\omega^2$.
Work. Fx foot poundals.	Work. $T\theta$. foot poundals.
Power. $F\dot{x}$ foot poundals per sec.	Power. $T\theta$ foot poundals per sec.

Simple Harmonic Motion.

In Fig. 19 let the radius vector OP rotate with uniform angular velocity α radians per second and make a complete revolution in time T. The point M, the projection of P on a diameter, will move to and fro with simple harmonic motion.

The maximum displacement of M from the mid-position O will be equal to OP, or a, the half amplitude of vibration. The displacement x from the mid-position at any time will be $a \sin \theta$ when θ is measured from the position of zero displacement. To make the expression perfectly general, and to permit the

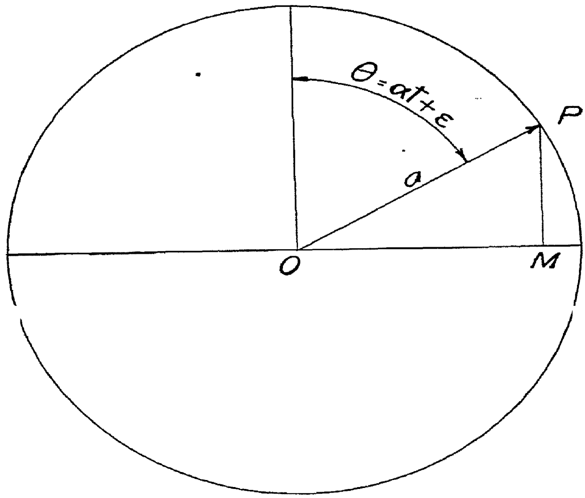


FIG. 19.

measurement of time from any instant, let θ have a value ε when $t = 0$. The value of θ at any instant t will be of the form $\alpha t + \varepsilon$, so that we may write

$$x = a \sin (\alpha t + \varepsilon).$$

For the velocity at any instant, differentiating with regard to t,

$$\dot{x} = a\alpha \cos (\alpha t + \varepsilon).$$

For the acceleration at any instant, differentiating again,

$$\ddot{x} = -a\alpha^2 \sin(\alpha t + \varepsilon).$$

Substituting x for $a \sin(\alpha t + \varepsilon)$,

we obtain $\ddot{x} = -\alpha^2 x.$

An alternative method of obtaining the same result is as follows:

$$\begin{aligned} x &= a \sin(\alpha t + \varepsilon) \\ &= a \cos \alpha t \sin \varepsilon + a \sin \alpha t \cos \varepsilon \\ &= \mathrm{A} \cos \alpha t + \mathrm{B} \sin \alpha t, \text{ where } \mathrm{A} = a \sin \varepsilon \text{ and } \mathrm{B} = a \cos \varepsilon, \end{aligned}$$

showing that due allowance for phase position when $t = 0$ may be made by expressing the displacement as the sum of cosine and sine functions.

As before, differentiating to obtain the velocity:

$$\dot{x} = -\mathrm{A}\alpha \sin \alpha t + \mathrm{B}\alpha \cos \alpha t.$$

Differentiating again to obtain the acceleration:

$$\ddot{x} = -\mathrm{A}\alpha^2 \cos \alpha t - \mathrm{B}\alpha^2 \sin \alpha t.$$

Substituting x for

$$\mathrm{A} \cos \alpha t + \mathrm{B} \sin \alpha t,$$

we find that $\ddot{x} = -\alpha^2 x$ as before.

This is the general differential equation for free simple harmonic motion, and whenever, in a given case, it can be established that the acceleration is proportional to the displacement but of opposite sign, the form may be recognized as that of simple harmonic motion, and numerical values may be substituted for α^2 in the above equations.

As applied to the solution of practical cases of vibration, α is the angular velocity in radians per second of an imaginary radius vector whose projection at any instant gives the value of the displacement from the position of rest. α may be termed the "circular frequency." In no case need there be any part of the actual system having an angular motion of this velocity.

The numerical value of α in any given case of vibration will be dependent on the ratio of the stiffness and the inertia of the vibrating system.

The time of oscillation T will be the same as that taken by a single revolution of the radius vector. But $x = a \sin(\alpha t + \varepsilon)$ and x has the same value after an interval T.

Therefore, $x = a \sin(\alpha\overline{t + \mathrm{T}} + \varepsilon)$.

The angle turned through in the time T $= \alpha\mathrm{T} = 2\pi$ radians. Therefore, if the radius vector is considered to have moved through a complete revolution in the time T, and to have come back to the same position :

$$\mathrm{T} = \frac{2\pi}{\alpha},$$

in which expression an appropriate numerical substitution for α will give the time of oscillation for the particular case.

Consider for a particular case that of a mass suspended by a helical spring.

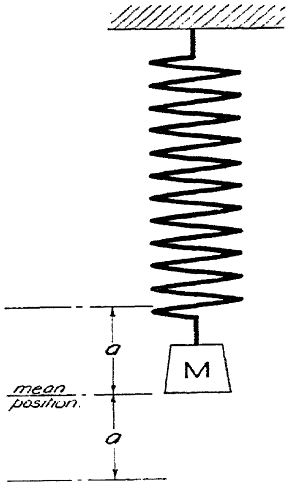


FIG. 20.

M $=$ mass in lbs.
f $=$ stiffness of the spring (in poundals per foot of deflection).
x $=$ displacement from mean position in feet.

Then at any instant the force exerted on the mass by the spring is $\mathrm{M}\ddot{x}$, which is equal in magnitude but of opposite sign to the force stretching the spring, viz., fx.

Therefore

$$\mathrm{M}\ddot{x} = -fx,$$

i.e.

$$\ddot{x} = -\frac{fx}{\mathrm{M}}.$$

This is immediately recognizable as being of the form $\ddot{x} = -\alpha^2 x$, in which $\alpha = \sqrt{\dfrac{f}{\mathrm{M}}}$.

Other cases of elastic vibration may be similarly treated, but,

for the moment, we shall continue the study of the helical spring under various conditions. The equations obtained will be applicable to all cases which follow the simple harmonic law, but it will assist thought if an easily conceived case, such as the helical spring, is studied throughout in all its bearings.

Making the substitution for this particular case

$$\ddot{x} = -\frac{fx}{\mathrm{M}} \text{ and } \alpha^2 = \frac{f}{\mathrm{M}}.$$

Therefore $$x = a \sin\left(\sqrt{\frac{f}{\mathrm{M}}} \,.\, t + \varepsilon\right).$$

And if $$t = 0 \text{ when } x = 0,$$

$$x = a \sin\left(\sqrt{\frac{f}{\mathrm{M}}} \,.\, t\right).$$

The time of oscillation,

$$\mathrm{T} = 2\pi \sqrt{\frac{\mathrm{M}}{f}}.$$

Damped Vibration.

Vibrations of the type just considered are called free vibrations, and are due to elastic forces only. Such vibrations are not met with in actual experience, there being invariably a force acting in opposition to the motion.

A resistance such as that due to air friction will vary as some variable function of the velocity, and for low velocities will be nearly as the first power. This will introduce a force $-\mu\dot{x}$ acting always in opposition to the motion and changing sign as the velocity changes in sign. The differential equation is then

$$\mathrm{M}\ddot{x} = -\mu\dot{x} - fx.$$

i.e. $$\ddot{x} + \frac{\mu\dot{x}}{\mathrm{M}} + \frac{fx}{\mathrm{M}} = 0,$$

Rewrite the expression more generally:

$$\ddot{x} + k\dot{x} + \alpha^2 x = 0,$$

where for the spring $k = \frac{\mu}{\mathrm{M}}$ and $\alpha^2 = \frac{f}{\mathrm{M}}$.

Consider first the case of simple harmonic motion without damping:

$$\ddot{x} + \alpha^2 x = 0,$$

i.e. $(D^2 + \alpha^2)\, x = 0$, where D stands for $\frac{d}{dt}$.

Making the substitution $x = e^{zt}$,

$$(z^2 + \alpha^2)e^{zt} = 0\text{; therefore } z^2 + \alpha^2 = 0$$

and $z = \pm i\sqrt{\alpha^2}$, where $i = \sqrt{-1}$.

The solution is
$$\begin{aligned} x &= C_1e^{z_1t} + C_2e^{z_2t} \\ &= C_1e^{i\alpha t} + C_2{}^{-i\alpha t}, \end{aligned}$$

in which C_1 and C_2 are constants of integration.

To determine their values put $\alpha t = \theta$. Then we have

$$x = C_1e^{i\theta} + C_2e^{-i\theta}.$$

But $e^{i\theta} = \cos\theta + i\sin\theta$, and $e^{-i\theta} = \cos\theta - i\sin\theta$.*

Therefore
$$\begin{aligned} x &= C_1(\cos\theta + i\sin\theta) + C_2(\cos\theta - i\sin\theta) \\ &= (C_1 + C_2)\cos\theta + i(C_1 - C_2)\sin\theta. \\ &= A\cos\theta + B\sin\theta. \\ &= A\cos at + B\sin\alpha t. \end{aligned}$$

But $A = a\sin\varepsilon$ and $B = a\cos\varepsilon$, ε being the phase angle when $t = 0$.

For the special case of the spring,

$$x = a\sin\left(\sqrt{\frac{f}{M}}\,.\,t + \varepsilon\right).$$

Returning to the general case of damped vibration for which

$$\ddot{x} + k\dot{x} + \alpha^2 x = 0 \quad \text{or} \quad (D^2 + kD + \alpha^2)x = 0,$$

making the substitution $x = e^{zt}$, we have

$$(z^2 + kz + \alpha^2)e^{zt} = 0.$$

Therefore the solution is $x = C_1e^{z_1t} + C_2e^{z_2t}$, where z_1 and z_2 are roots of the equation $z^2 + kz + \alpha^2 = 0$.

There are four cases to be considered:

(1) *Where* $k = 0$, *i.e.* resistance is negligible. Then the solution is, as already shown,

$$x = a\sin(\alpha t + \varepsilon) \text{ and } T = \frac{2\pi}{\alpha}.$$

* See Appendix.

(2) *When the resistance is large* and $k^2 > 4\alpha^2$.

Then
$$z = \frac{-k \pm \sqrt{k^2 - 4\alpha^2}}{2}.$$

By hypothesis, the roots z_1 and z_2 are real. Since their product is positive, they have the same sign, which is negative.

The solution is
$$x = C_1 e^{z_1 t} + C_2 e^{z_2 t}.$$

If it is known that when $t = 0$, $x = a$ and $\dot{x} = 0$, C_1 and C_2 may be calculated.

For
$$a = C_1 + C_2\,; \text{ and } 0 = z_1 C_1 + z_2 C_2.$$

In this case it is found that the displacement falls asymptotically to zero after starting at its maximum value. The motion is not of an oscillatory character.

(3) *When resistance is large* and $k^2 = 4\alpha^2$.

Then
$$z = \frac{-k \pm \sqrt{k^2 - 4\alpha^2}}{2} = -\frac{k}{2} \pm 0.$$

Then the solution is
$$\begin{aligned} x &= C_1 e^{z_1 t} + C_2 e^{z_2 t} \\ &= (C_1 + C_2) e^{zt}. \end{aligned}$$

Here we have but one arbitrary constant and the method fails. Assume the solution to be of the usual form, h in what follows being infinitely small. Then

$$\begin{aligned} x &= C_1 e^{zt} + C_2 e^{(z+h)t} = e^{zt}(C_1 + C_2 e^{ht}) \\ &= e^{zt}\left\{C_1 + C_2\left(1 + ht + \frac{h^2t^2}{\lfloor 2} + \frac{h^3t^3}{\lfloor 3} + \ldots\right)\right\} \\ &= e^{zt}\left\{C_1 + C_2 + C_2 ht\left(1 + \frac{ht}{\lfloor 2} + h^2t^2 + \ldots\right)\right\}. \end{aligned}$$

As h is very small,
$$\begin{aligned} x &= e^{zt}(C_1 + C_2) + e^{zt} C_2 ht \\ &= e^{zt}(A + Bt), \end{aligned}$$

which gives the solution
$$x = e^{\frac{-kt}{2}}(A + Bt).$$

In this case there is again no oscillatory motion. The first

factor diminishes with the time t, while the second factor increases, but at a slower rate.

(4) *When the resistance is small* and $k^2 < 4\alpha^2$.

Then $$z = -\frac{k \pm i\sqrt{4\alpha^2 - k^2}}{2}.$$

The roots in this case are therefore imaginary, viz.:

$$z_1 = -\frac{k}{2} + i\sqrt{\alpha^2 - \frac{k^2}{4}} \text{ and } z_2 = -\frac{k}{2} - i\sqrt{\alpha^2 - \frac{k^2}{4}}.$$

The solution is: $x = C_1 e^{z_1 t} +$

$$= e^{-\frac{k}{2}t}\left(C_1 e^{i\sqrt{\alpha^2 - \frac{k^2}{4}}\,.\,t} + C_2 e^{-i\sqrt{\alpha^2 - \frac{k^2}{4}}\,.\,t}.\right.$$

The quantity within the brackets may be dealt with exactly as in the previously considered case of undamped vibration θ in this case being $\sqrt{\alpha^2 - \frac{k^2}{4}}\,.\,t$.

Then $x = e^{-\frac{k}{2}t}(A \cos \theta + B \sin \theta)$

$$= e^{-\frac{k}{2}t}\left(A \cos \sqrt{\alpha^2 - \frac{k^2}{4}}\,.\,t + B \sin \sqrt{\alpha^2 - \frac{k^2}{4}}\,.\,t\right)$$

$$= e^{-\frac{k}{2}t} a\,.\,\sin\left(\sqrt{\alpha^2 - \frac{k^2}{4}}\,.\,t + \varepsilon\right).$$

This represents a simple harmonic motion for which

$$T = \frac{2\pi}{\sqrt{\alpha^2 - \frac{k^2}{4}}},$$

and the amplitude of which diminishes asymptotically to zero, the diminution being expressed by the factor $e^{-\frac{k}{2}\,.\,t}$.

Numerical Example of Damped Vibration of Loaded Spring.

The equation of motion is:

$$\ddot{x} + \frac{\mu\dot{x}}{M} + \frac{fx}{M} = 0,$$

where M = mass in pounds, f = stiffness in poundals per foot,

and μ = coefficient of friction in poundals per unit velocity. Or more generally: $\ddot{x} + k\dot{x} + \alpha^2 x = 0$, where $k = \frac{\mu}{M}$ and $\alpha^2 = \frac{f}{M}$.

(1) *The case of undamped vibration* has as solution

$$x = a \sin(\alpha t + \varepsilon) = a \sin\left(\sqrt{\frac{f}{M}} \cdot t + \varepsilon\right).$$

If $M = 4$ and $f = 169$

$$x = a \sin\left(\frac{13}{2}t + \varepsilon\right).$$

For the three cases of damped vibration we have:

(2) *When resistance is large* and $k^2 > 4a^2$.

Let μ be 340 and $k = 85$.

The solution is $\qquad x = C_1 e^{z_1 t} + C_2 e^{z_2 t}$,

where $\quad z = -85 \pm \sqrt{85^2 - 4 \times \frac{169}{4}} = -\frac{169}{2}$ and $-\frac{1}{2}$.

Assuming the time to be measured in such manner that when $t = 0$, $x = a$ and $\dot{x} = 0$ we have:

$$x = C_1 e^{-\frac{1}{2}t} + C_2 e^{-\frac{169}{2}t},$$

and therefore

$$\dot{x} = -\frac{1}{2}C_1 e^{-\frac{1}{2}t} - \frac{169}{2}C_2{}^{-\frac{169}{2}t}.$$

From which

$$a = C_1 + C_2, \text{ and } 0 = -\frac{1}{2}C_1 - \frac{169}{2}C_2,$$

so that $\qquad C_1 = \frac{169}{168}a$ and $C_2 = -\frac{a}{168}$.

Therefore $\qquad x = \frac{169}{168}ae^{-\frac{1}{2}t} - \frac{a}{168}e^{-\frac{169}{2}t}$.

(3) *When the resistance is large* and $k^2 = 4\alpha^2$, *i.e.* when $\mu = 52$ and $k = 13$.

Then $\qquad x = e^{-\frac{k}{2}t}(A + Bt)$,

and $\qquad \dot{x} = e^{-\frac{k}{2}t}\left(B - \frac{Btk}{2} - \frac{Ak}{2}\right)$.

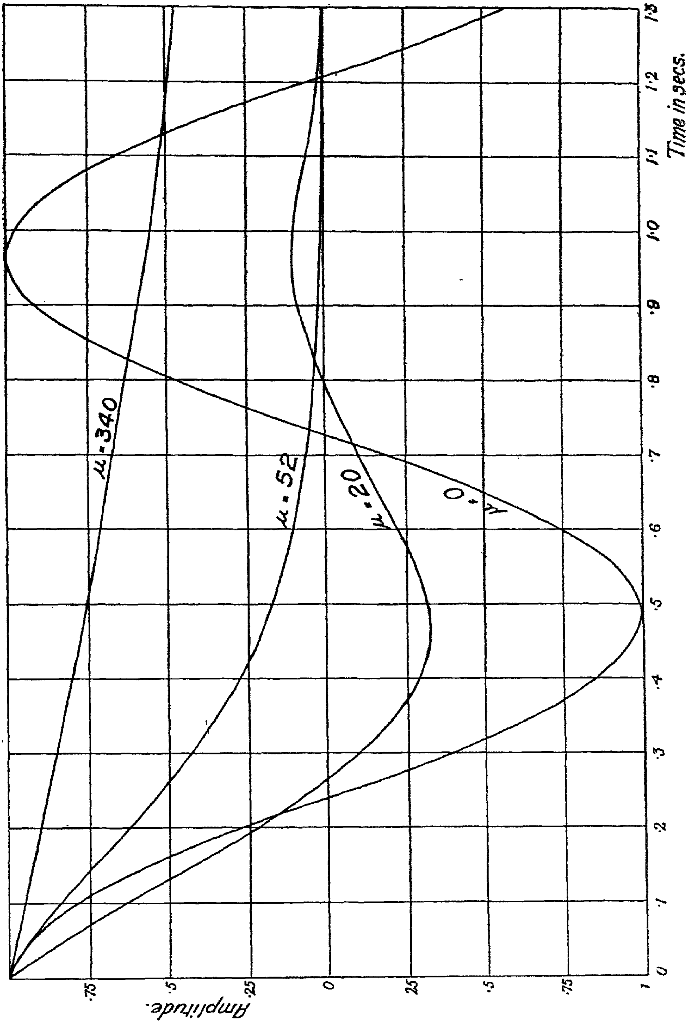

Time in secs.
0
·1
·2
·3
·4
·5
·6
·7
·8
·9
1·0
1·1
1·2
1·3
Amplitude.
·75
·5
·25
0
·25
·5
·75
1
μ = 340
μ = 52
μ = 20
μ = 0

FIG. 21.

Again, when $t = 0$, $x = a$, and $\dot{x} = 0$.

Therefore $$a = \mathrm{A} \text{ and } 0 = \mathrm{B} - \frac{\mathrm{A}k}{2}.$$

From which $$\mathrm{A} = a \text{ and } \mathrm{B} = \frac{ak}{2}.$$

Hence we have
$$x = e^{-\frac{k}{2}t}a\left(1 + \frac{k}{2}t.\right) = e^{-\frac{13}{2}.t}a\left(1 + \frac{13}{2}t\right).$$

(4) *When the resistance is small* and $k^2 < 4\alpha^2$.
Let $\mu = 20$ and $k = 5$.
Then
$$x = e^{-\frac{k}{2}t}a \sin\left(\sqrt{\alpha^2 - \frac{k^2}{4}}.t + \varepsilon\right) = e^{-\frac{5}{2}t}a \sin(6t + \varepsilon).$$

The four cases are plotted in the accompanying Fig. 21 for a value of a of unity.

It will be seen that cases (2) and (3) do not give oscillating motions.

The time of oscillation for case (1) with no damping is
$$\frac{2\pi}{\alpha} = \frac{4\pi}{13},$$

and for case (4), with damping, it is increased to
$$\frac{2\pi}{\sqrt{\alpha^2 - \frac{k^2}{4}}} = \frac{4\pi}{12}.$$

Forced Vibrations.

When a system such as has been already considered, consisting of inertia and elasticity, its motion being subject to a resistance, such as friction, varying as the velocity, is acted upon by a periodic disturbing force, the resultant motion is a forced vibration. Consider the case in which the periodic disturbing force is a simple harmonic function of the time.

Let the force be $\mathrm{P}\cos(\omega t + \varepsilon')$, where P is its maximum value. The equation of motion for the spring is
$$\mathrm{M}\ddot{x} + \mu\dot{x} + fx = \mathrm{P}\cos(\omega t + \varepsilon').$$

Let $p = \frac{\mathrm{F}}{\mathrm{M}}$, and the more general equation becomes:

$$\ddot{x} + k\dot{x} + \alpha^2 x = p\cos(\omega t + \varepsilon').$$

The complete solution is $x = x_1 + x_2$, where x_1, the particular integral, is any function which satisfies the equation as it stands, and x_2 the complementary function, which is the general solution of the equation

$$\ddot{x} + k\dot{x} + \alpha^2 x = 0.$$

For the particular integral assume

$$x_1 = \mathrm{A}'\cos(\omega t + \varepsilon') + \mathrm{B}'\sin(\omega t + \varepsilon').$$

Then $\quad \dot{x}_1 = -\mathrm{A}'\omega\sin(\omega t + \varepsilon') + \mathrm{B}'\cos(\omega t + \varepsilon'),$

and $\quad \ddot{x}_1 = -\mathrm{A}'\omega^2\cos(\omega t + \varepsilon') - \mathrm{B}'\omega^2\sin(\omega t + \varepsilon').$

Substituting in the general equation of motion we find

$$(\alpha^2\mathrm{A}' + k\omega\mathrm{B}' - \mathrm{A}'\omega^2)\cos(\omega t + \varepsilon') + (\alpha^2\mathrm{B}' - k\omega\mathrm{A}' - \mathrm{B}'\omega^2)\sin(\omega t + \varepsilon') = p\cos(\omega t + \varepsilon'),$$

i.e. $$\mathrm{A}'(\alpha^2 - \omega^2) - \mathrm{A}'k\omega \cdot \frac{\sin(\omega t + \varepsilon')}{\cos(\omega t + \varepsilon')} + \mathrm{B}'(\alpha^2 - \omega^2)\frac{\sin(\omega t + \varepsilon')}{\cos(\omega t + \varepsilon')} + \mathrm{B}'k\omega = p.$$

Putting $(\omega t + \varepsilon') = 0$, we see that $\mathrm{A}'(\alpha^2 - \omega^2) + \mathrm{B}'k\omega = p$, which will have this value for all values of $(\omega t + \varepsilon')$. Hence $-\mathrm{A}'k\omega + \mathrm{B}'(\alpha^2 - \omega^2) = 0$. Solving for A' and B',

$$\mathrm{A}' = \frac{\alpha^2 - \omega^2}{(\alpha^2 - \omega^2)^2 + k^2\omega^2}p \text{ and } \mathrm{B}' = \frac{k\omega}{(\alpha^2 - \omega^2)^2 + k^2\omega^2}p.$$

The particular integral is therefore

$$x_1 = \mathrm{A}'\cos(\omega t + \varepsilon') + \mathrm{B}'\sin(\omega t + \varepsilon')$$

where A' and B' have the above values.

The complementary function is the solution of

$$\ddot{x} + k\dot{x} + \alpha^2 x = 0.$$

i.e. $$x_2 = e^{-\frac{k}{2}t}(\mathrm{A}\cos\alpha' t + \mathrm{B}\sin\alpha' t)$$

$$= e^{-\frac{k}{2}t}a(\sin\alpha' t + \varepsilon),$$

where $$\alpha' = \sqrt{\alpha^2 - \frac{k}{4}}.$$

The complete solution is therefore :

$$x = x_1 + x_2$$

$$= e^{-\frac{k}{2}t}(\text{A} \cos \alpha' t + \text{B} \sin \alpha' t) + (\text{A}' \cos \overline{\omega' t + \varepsilon'} + \text{B}' \sin \overline{\omega t + \varepsilon'}).$$

(natural) (forced)

The first expression on the right-hand side of the equation represents the natural oscillation, and the second, the forced oscillation.

The natural vibration, under the influence of the viscous resistance, will eventually become zero, leaving only the forced vibration.

The motion then settles down to the state defined by the equation

$$x = 0 + \text{A}' \cos (\omega t + \varepsilon') + \text{B}' \sin (\omega t + \varepsilon')$$

$$= \frac{\alpha^2 - \omega^2}{(\alpha^2 - \omega^2)^2 + k^2\omega^2} p \,.\cos (\omega t + \varepsilon') + \frac{k\omega}{(\alpha^2 - \omega^2)^2 + k^2\omega^2} p \,.\sin (\omega t + \varepsilon).$$

When friction is small k tends to zero, and we have

$$x = \frac{\alpha^2 - \omega^2}{(\alpha^2 - \omega^2)^2} p \,.\cos (\omega t + \varepsilon') = \frac{p}{\alpha^2 - \omega^2} \cos (\omega t + \varepsilon').$$

This is a simple harmonic motion of the same period as the disturbing force, and in phase with it, the amplitude being

$$\frac{p}{\alpha^2 - \omega^2}.$$

This may be written $\quad \frac{\alpha^2}{\alpha^2 - \omega^2} \times \frac{p}{\alpha^2}.$

It will be seen by reference to a loaded spring for which

$$p = \frac{\text{P}}{\text{M}} \text{ and } \alpha^2 = \frac{f}{\text{M}} \text{ that } \frac{p}{\alpha^2} = \frac{\text{P}}{\text{M}} \times \frac{\text{M}}{f} = \frac{\text{P}}{f},$$

which is the static deflection of the spring corresponding to the maximum value of the disturbing force.

The actual amplitude of the vibration $= \frac{\quad}{\alpha^2 - \omega^2} \times \frac{p}{\alpha^2}.$

Hence the fraction

$$\frac{\text{Amplitude of Vibration}}{\begin{array}{c}\text{(Static deflection due to maximum}\\ \text{value of the disturbing force)}\end{array}} = \frac{\alpha^2}{\alpha^2 - \omega^2} = \frac{1}{1 - \left(\frac{\omega}{\alpha}\right)^2}.$$

This fraction gives the degree of amplification, and may be called the "amplification factor."

Reverting to the equation for the forced vibration

$$x = \frac{\alpha^2 - \omega^2}{(\alpha^2 - \omega^2)^2 + k^2\omega^2}p \cos(\omega t + \varepsilon') + \frac{k\omega}{(\alpha^2 - \omega^2)^2 + k^2\omega^2}p \sin(\omega t + \varepsilon').$$

When $\omega = \alpha$ the condition for resonance, the first term vanishes, and

$$x = 0 + \frac{p}{k\omega} . \sin(\omega t + \varepsilon').$$

As k is generally very small, we again see that the amplitude becomes very great. The phase is 90° behind the disturbing force, which is proportional to $p(\cos \omega t + \varepsilon')$.

If the frequency of the disturbing force is much greater than the natural frequency, *i.e.* if ω is much greater than α, and k is small, the first term becomes of importance and

$$x = -\frac{p}{\omega^2} . \cos(\omega t + \varepsilon') \text{ approximately.}$$

The amplitude has become small again, and the phase is opposite to that of the disturbing force

The expression

$$x = \frac{\alpha^2 - \omega^2}{(\alpha^2 - \omega^2)^2 + k^2\omega^2}p \cos(\omega t + \varepsilon') + \frac{k\omega}{(\alpha^2 - \omega^2)^2 + k^2\omega^2}p . \sin(\omega t + \varepsilon')$$

may be used for calculating, not only x, but also the change of phase, as the relation between ω and α varies. It is of the form

$$x = A' \cos(\omega t + \varepsilon') + B' \sin(\omega t + \varepsilon'),$$

in which
$$A' = \frac{\alpha^2 - \omega^2}{(\alpha^2 - \omega^2)^2 + k^2\omega^2}p,$$

and
$$B' = \frac{k\omega}{(\alpha^2 - \omega^2)^2 + k^2\omega^2}p.$$

But x may also be expressed in the form

$$x = b \cos (\omega t + \varepsilon' - \phi),$$

where b is the half amplitude of the forced vibration, and ϕ is the phase difference between the disturbing force and the resultant displacement. Therefore

$$\begin{aligned} x &= b \cos (\omega t + \varepsilon') \cos \phi + b \sin (\omega t + \varepsilon') \sin \phi \\ &= b \cos \phi \cos (\omega t + \varepsilon') + b \sin \phi \sin (\omega t + \varepsilon') \\ &= A' \cos (\omega t + \varepsilon') + B' \sin (\omega t + \varepsilon'). \end{aligned}$$

Therefore $\quad A'^2 + B'^2 = b^2 \cos^2 \phi + b^2 \sin^2 \phi = b^2,$

or
$$b = \sqrt{A'^2 + B'^2} = \sqrt{\frac{(\alpha^2 - \omega^2)^2 p^2 + k^2\omega^2 p^2}{\{(\alpha^2 - \omega^2)^2 + k^2\omega^2\}^2}}$$
$$= \frac{p}{\sqrt{(\alpha^2 - \omega^2)^2 + k^2\omega^2}}.$$

Now
$$\tan \phi = \frac{\sin \phi}{\cos \phi} = \frac{B'}{A'} = \frac{k\omega}{(\alpha^2 - \omega^2)}.$$

The above two expressions enable the amplitude of a forced vibration and the angle by which the displacement lags behind the disturbing force to be readily calculated when the general equation of motion is

$$\ddot{x} + k\dot{x} + \alpha^2 x = p \cos (\omega t + \varepsilon').$$

Now we found that the equation for a loaded spring, acted upon by a disturbing force of maximum value P, was

$$M\ddot{x} + \mu\dot{x} + fx = P \cos (\omega t + \varepsilon'),$$

or
$$\ddot{x} + \frac{\mu}{M}\dot{x} + \frac{f}{M}x = \frac{P}{M} \cos (\omega t + \varepsilon').$$

In order to calculate the amplitude and phase for this special case, we must substitute $\frac{\mu}{M}$ for k, $\frac{f}{M}$ for α^2 and $\frac{P}{M}$ for p.

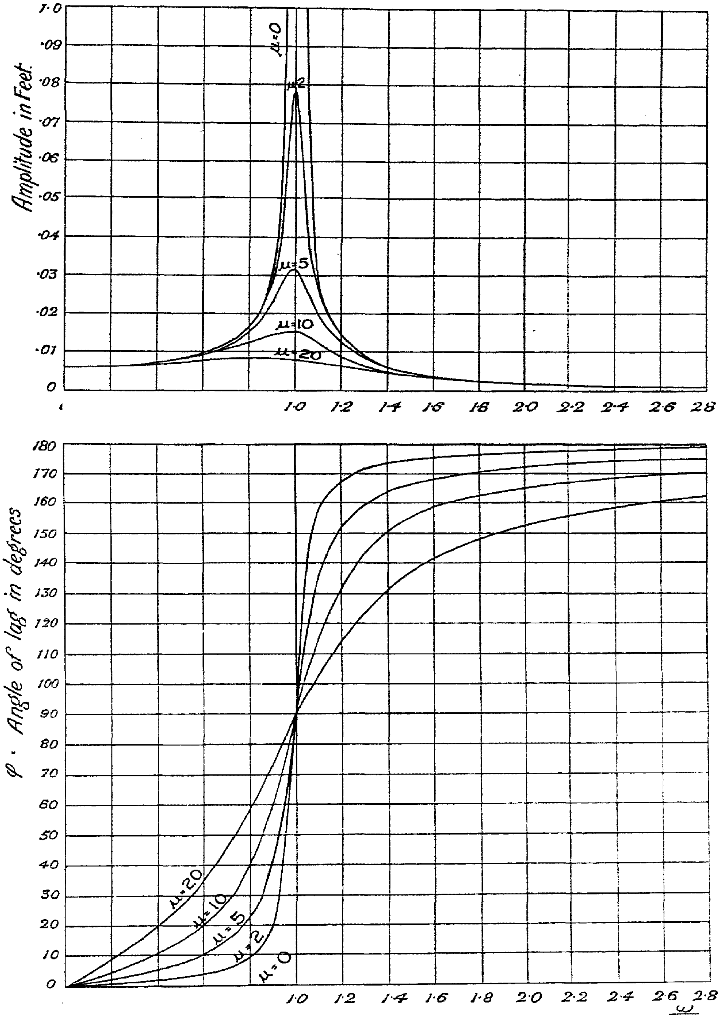

Amplitude in Feet.
1·0
·09
·08
·07
·06
·05
·04
·03
·02
·01
0
μ=0
μ=2
μ=5
μ=10
μ=20
1·0 1·2 1·4 1·6 1·8 2·0 2·2 2·4 2·6 2·8
φ · Angle of lag in degrees
180
170
160
150
140
130
120
110
100
90
80
70
60
50
40
30
20
10
0
μ=20
μ=10
μ=5
μ=2
μ=0
1·0 1·2 1·4 1·6 1·8 2·0 2·2 2·4 2·6 2·8
ω

FIG. 22.

We then have

$$b=\frac{\frac{\mathrm{P}}{\mathrm{M}}}{\sqrt{\left(\frac{f}{\mathrm{M}}-\omega^2\right)^2+\left(\frac{\mu}{\mathrm{M}}\omega\right)^2}}=\frac{\mathrm{P}}{\sqrt{\mathrm{M}^2\left(\frac{f}{\mathrm{M}}-\omega^2\right)^2+\mu^2\omega^2}}$$

and
$$\tan\phi=\frac{\mu\omega}{f-\mathrm{M}\omega^2}$$

Fig. 22 shows the amplitude and phase change for the particular case of the loaded spring already considered, when subjected to a harmonically varying disturbing force the maximum value of which is 1 poundal. The curves are drawn for various values of μ.

A Whirling Shaft considered as a Case of Forced Vibration.

This is dealt with exhaustively in Part I. It is here proposed to show how the elastic deflection for any given speed of rotation may be derived from the general expression for forced vibration, viz.:

$$\ddot{x}+k\dot{x}+\alpha^2=p\cos(\omega t+\varepsilon').$$

Consider a simple case of a mass M pounds carried on a very light shaft with an initial eccentricity r.

Let f be the stiffness of the shaft in poundals per foot of deflection, and μ the coefficient of friction in poundals.

The disturbing force is due to the initial eccentricity, and its maximum value is a function of the speed of rotation, and is $\mathrm{M}\omega^2 r$. The whirling motion is to be regarded as that due to two vibrations at right angles, for each of which the equation is

$$\mathrm{M}\ddot{x}+\mu\dot{x}+fx=\mathrm{M}\omega^2 r\cos(\omega t+\varepsilon'),$$

or
$$\ddot{x}+\frac{\mu}{\mathrm{M}}x+\frac{f}{\mathrm{M}}x=\omega^2 r\cos(\omega t+\varepsilon').$$

In order to obtain the amplitude and phase for this special case, we must substitute in the general equations $\frac{\mu}{\mathrm{M}}$ for k, $\frac{f}{\mathrm{M}}$ for α^2 and $\omega^2 r$ for p. The maximum elastic deflection is

$\dfrac{\omega^2 r}{\sqrt{\left(\frac{f}{M} - \omega^2\right)^2 + \left(\frac{\mu}{M}\right)^2 \omega^2}}$, which lags behind the initial eccentricity by an angle whose tangent is $\dfrac{\mu\omega}{f - M\omega^2}$.

The Angular Acceleration or Rate of Change of Angular Momentum of a Rotating Mass about a Fixed Axis is Proportional to the External Torque.

Let a body of a total mass M be rotated about an axis O by the application of a turning moment T. Consider a small

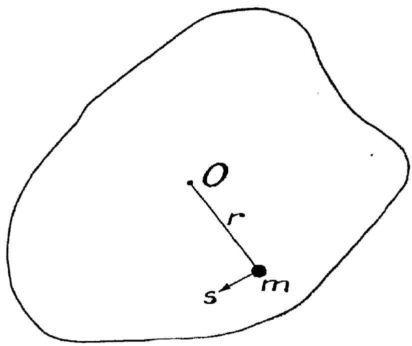


FIG. 23.

element of the body of mass m at radius r from the axis, and let its displacement in the direction of motion be s. Then the force acting on m will be $m\ddot{s}$.

But $$s = r\theta \text{ and } \ddot{s} = r\ddot{\theta}.$$

Therefore the force acting on m is $mr\ddot{\theta}$, and moment of the force about O is $mr^2\ddot{\theta}$. The sum of all such moments acting on the elements composing the body will be equal to the applied torque. We may write, therefore,

$$T = \Sigma mr^2\ddot{\theta} = Mk^2\ddot{\theta},$$

i.e. the applied torque is equal to the rate of change of the angular momentum.

The Simple Pendulum.

Let a mass M be suspended from a point O on a string of length l. When displaced through an angle θ from the position of rest the restoring moment is $Mgl \sin \theta$, or if the displacement is small we may write it $Mgl\theta$, which is of equal magnitude but

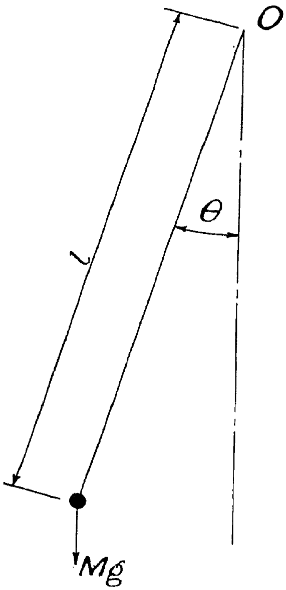


FIG. 24.

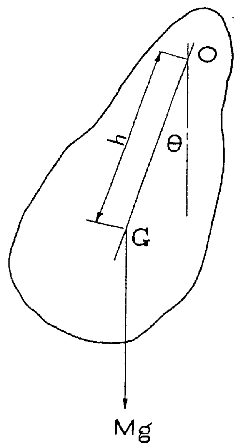


FIG. 25.

opposite sign to the rate of change of angular momentum. Therefore

$$Mk^2\ddot{\theta} = -Mgl\theta, \text{ or } \ddot{\theta} = -\frac{g}{l}\theta,$$

which is of the form $\ddot{x} = -\alpha^2 x,$

so that $\theta = a \sin\left(\sqrt{\frac{g}{l}} \cdot t + \varepsilon\right),$

and the time of oscillation is $2\pi\sqrt{\frac{l}{g}}.$

The Compound Pendulum.

Let any non-symmetrical body of mass M be suspended from a point O distant h from its centre of gravity G. When displaced through an angle θ from its position of rest the restoring moment is $\mathrm{M}gh \sin \theta$, which is of equal magnitude but opposite sign to the rate of change of angular momentum.

Therefore $\qquad \mathrm{M}k^2\ddot{\theta} = -\mathrm{M}gh \sin \theta$, or

$$\ddot{\theta} = -\frac{gh}{k^2}\,\theta, \text{ if } \theta \text{ be very small.}$$

This equation is of the form $\ddot{x} = -\alpha^2 x$.

So that $$\theta = a \sin\left(\sqrt{\frac{gh}{k^2}} \cdot t + \varepsilon\right),$$

and the time of oscillation is $\quad 2\pi\sqrt{\dfrac{k^2}{gh}}$.

The value of the moment of inertia of the body about its centre of gravity G may be found from the above. For if k be the radius of gyration about O, $\bar{k}$ the radius of gyration about G, and h the distance between O and G, then $k^2 = \bar{k}^2 + h^2$.
Thus the time of oscillation of the compound pendulum about O will be

$$2\pi\sqrt{\frac{\bar{k}^2 + h^2}{gh}}.$$

Now if the length of a simple pendulum is adjusted until it oscillates in synchronism with the compound pendulum, we may equate the two expressions for the periodic time thus—

$$2\pi\sqrt{\frac{\bar{k}^2 + h^2}{gh}} = 2\pi\sqrt{\frac{l}{g}},$$

whence $$l = \frac{\bar{k}^2 + h^2}{h}, \text{ or}$$
$$\bar{k}^2 = lh - h^2.$$

Multiplication by the mass M gives the moment of inertia about the mass centre.

Bifilar and Trifilar Suspension.

Bifilar Suspension—If a bar of mass M, suspended by two wires of length l, the points of support being distant $2d$, is set into torsional vibration, the period of oscillation may be found as follows.

The vertical force at the end of each wire is $\frac{M}{2}g$. If at any instant the bar is displaced through a small angle θ, and the suspending wire makes an angle ϕ with the vertical, we have $d\theta = l\phi$.

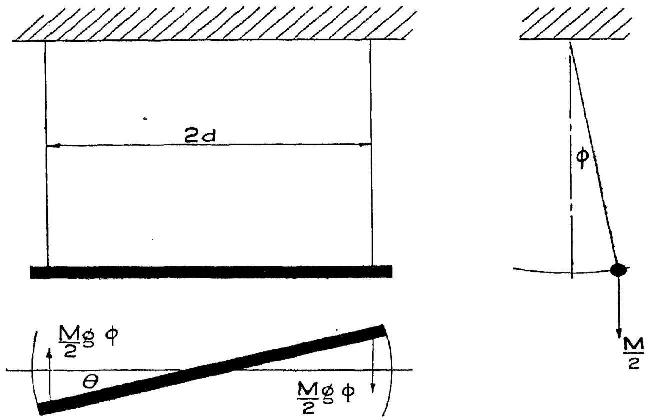


FIG. 26.

The restoring moment about the point of suspension will be, therefore, $\frac{M}{2}gl\phi$ as in the case of a simple pendulum, and the horizontal restoring force, acting on each end of the bar, will be $\frac{M}{2}g\phi$.

Hence the moment of the couple tending to restore equilibrium will be

$$\frac{M}{2}g\phi \times 2d,$$

or putting

$$\phi = \frac{d}{l}\theta,$$

the restoring moment will be $\frac{Mgd^2\theta}{l}$,

so that
$$Mk^2\ddot{\theta} = -\frac{Mgd^2\theta}{l},$$

or
$$\ddot{\theta} = -\frac{gd^2\theta}{lk^2}.$$

This is of the form $\ddot{x} = -\alpha^2 x$, so that the time of oscillation is

$$2\pi\sqrt{\frac{lk^2}{gd^2}}.$$

Trifilar Suspension.—Similarly, if a mass be suspended symmetrically by three wires, the vertical force at the end of each wire will be $\frac{Mg}{3}$, and the horizontal restoring force acting at the point of attachment to the mass will be $\frac{Mg\phi}{3}$.

Let r be the distance of the point of attachment from the centre of oscillation. Then the moment exerted by this force is $\frac{Mg\phi r}{3}$ and the total moment exerted by the three such forces is $Mg\phi r$, or putting $\phi = \frac{r\theta}{l}$, the restoring moment is $\frac{Mgr^2\theta}{l}$.

Hence
$$Mk^2\ddot{\theta} = -\frac{Mgr^2\theta}{l},$$

or
$$\quad = -\frac{gr^2\theta}{lk^2},$$

which again is of the form $\ddot{x} = -\alpha^2 x$,
and the time of oscillation is

$$2\pi\sqrt{\frac{lk^2}{gr^2}}.$$

In either case of bifilar or trifilar suspension, if the time of oscillation is observed experimentally, the radius of gyration of the suspended mass may be calculated and the moment of inertia found.

To find the Moment of Inertia of a Body by Trifilar Suspension.

It may not be convenient to suspend the given body itself by three wires. In such a case a circular disc is so suspended, and the time of oscillation observed.

Then $$T_0 = 2\pi\sqrt{\frac{lk^2_0}{gr^2}},$$

where k_0 is the radius of gyration of the disc. The body whose moment of inertia is desired is then placed symmetrically on the disc, and the time of oscillation of the combination observed, say T_1, then

$$T_1 = 2\pi\sqrt{\frac{lk_1^2}{gr^2}}$$

where k_1 is the radius of gyration of the combination.

Then if m be the mass of the disc, M that of the given body, and k the radius of gyration of the given body, we have

$$mk_0^2 + Mk^2 = (m + M)k_1^2,$$

whence $$Mk^2 = (m + M)k_1^2 - mk_0^2.$$

Elastic Constants, etc.

Coefficient of Elasticity.—If an elastic material is stretched or compressed, the deformation is found to be proportional to the applied load. This is embodied in Hooke's law, which says that the radio of stress to strain is a constant, denoted usually by the symbol E.

Stress is defined as intensity of loading, in appropriate units. Usually pounds weight per square inch are the units used, but for the purpose of vibration problems confusion will be avoided by expressing stress in poundals per square foot. The symbol used for stress will be f.

Strain is the ratio of the extension or compression to the original length, and is denoted by e.

Expressing the law in these symbols we have

$$E = \frac{f}{e}.$$

This is the law which governs the elastic bending of a beam, the top fibres being in compression and the bottom in tension.

E will be in the same units as f, and will be the stress which would suffice to double the length of the material were it possible for it to remain elastic throughout the process.

Coefficient of Rigidity.—Similarly, if an elastic shaft is subjected to torque, the material is subject to a shear stress and strain, and again these two quantities are proportional one to the other.

Consider the opposite sides of a piece of elastic material of thickness h to be stressed by a shear stress of intensity q. Under this load one surface will be displaced with reference to the other, say through a distance d. Shear strain is defined as the ratio $\frac{d}{h}$, which is, of course, constant for a given stress, whatever the value of h. If the displacement is small, $\frac{d}{h}$ will be equal to the angle ϕ, expressed in circular measure, and this symbol is used for shear strain.

Again, the ratio of shear stress to shear strain is a constant, and substituting the appropriate symbols

$$\frac{q}{\phi} = C.$$

C is known as the *coefficient of rigidity*. Like E, it has the dimensions of a stress, and if q is taken in poundals per square foot C will be similarly expressed.

Torsional Vibration of Shafts

Torsion of Elastic Shafts.

Let a shaft be subjected to a torque T, and let the relative displacement of sections separated by a length l be θ radians. Consider an element of area a on one of those sections at a radius r. Let the intensity of shear stress on it be q.

The actual distance moved through by a in the direction of the strain is $r\theta$, which corresponds to d in the previous argument. The distance l between the two surfaces under consideration corresponds to h.

Therefore the strain $= \phi = \frac{r\theta}{l}$. The stress $= q$, and $C = \frac{q}{\phi}$, so that

$$q = C\phi = \frac{Cr\theta}{l}.$$

The torque represented by the forces on the element a is $aqr = \frac{ar^2C\theta}{l}$ and the total torque $= \frac{\Sigma ar^2C \,.\, \theta}{l}$. But $\Sigma ar^2 = I_0, =$ the second moment of area. Hence the torque is $\frac{I_0C\theta}{l}$, $\frac{I_0C}{l}$ being the torque necessary to twist the shaft through unit angle.

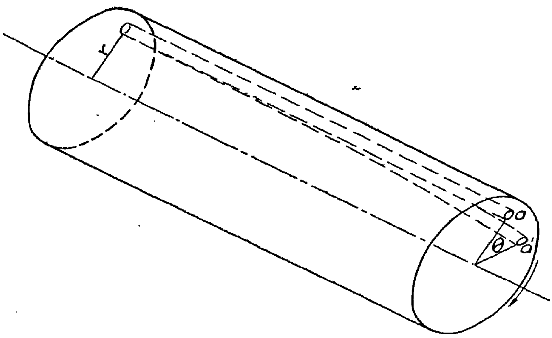


Fig. 27.

This expression will give the value of the restoring torque called up in the shaft in cases of torsional oscillation, in terms of θ the angular displacement, and will be equal and opposite to the torque applied by the oscillating mass.

It has been shown previously that the torque due to the oscillating mass is $Mk^2\ddot{\theta}$, where M is the mass and k is the radius of gyration.

We may write therefore

$$Mk^2\ddot{\theta} = -\frac{CI_0\theta}{l} \text{ or } \ddot{\theta} = -\frac{CI_0\theta}{Mk^2l},$$

which is of the general form for simple harmonic motion, viz. :—

$$\ddot{x} = -\alpha^2 x$$

in which $\alpha =$

Therefore $$= \frac{2\pi}{\alpha} = 2\pi\sqrt{\frac{Mk^2l}{CI_0}}.$$

Where M = mass in pounds.
k = radius of gyration in feet.
l = length in feet.
C = coefficient of rigidity in poundals per sq. ft.
I_0 = second moment of section in (feet)^4.

Single Mass on a Composite Shaft fixed at one end.

If the shaft is made up of several parts of differing lengths and second moments of area, as in Fig. 28, each constituent portion

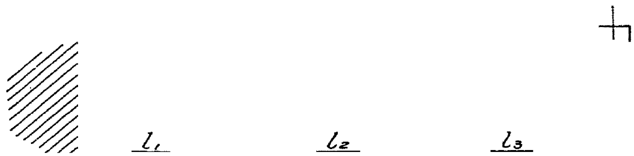


FIG. 28.

will twist through an angle dependent on its individual stiffness, and the total angle turned through will be the sum of these.

Calling the separate lengths l_1, l_2, l_3, etc., the second moments I_1, I_2, I_3, etc., and the angles θ_1, θ_2, θ_3, etc.

$$\theta_1 = \frac{l_1 T}{I_1 C}, \ \theta_2 = \frac{l_2 T}{I_2 C}, \ \theta_3 = \frac{l_3 T}{I_3 C}, \text{ etc.,}$$

where T is the applied torque.

The total angle turned through, viz.:

$$\theta = \Sigma \frac{lT}{IC}$$

and $$T = \frac{C\theta}{\Sigma \frac{l}{I}}.$$

Therefore $$Mk^2\ddot{\theta} = -\frac{C\theta}{\Sigma\frac{l}{I}}$$

and $$\alpha = \sqrt{\frac{C}{Mk^2\Sigma\frac{l}{I}}}.$$

Single Mass on a Composite Shaft fixed at both ends.

Supposing a single mass to be mounted as in Fig. 29. Let l_1 and l_2 be the lengths of the portions of the shaft, and I_1 and I_2 the second moments of area.

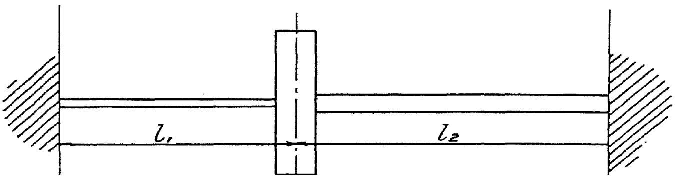


FIG. 29.

In this case the total restoring torque will be the sum of the torques exerted by the two shafts and will be equal to

$$\left(\frac{I_1}{l_1} + \frac{I}{l_2}\right)C\theta.$$

We then have $$Mk^2\ddot{\theta} = -\left(\frac{I_1}{l_1} + \frac{I_2}{l_2}\right)C\theta$$

and $$\alpha = \sqrt{\frac{C}{Mk^2}\left(\frac{I_1}{l_1} + \frac{I_2}{l_2}\right)}.$$

Two Masses on a Free-ended Shaft.

If two masses are arranged on a shaft as in Fig. 30 and are set oscillating in opposite directions, somewhere between the two masses there will be a node or stationary point which will divide the system into two single load systems of the type first considered, which will have the same frequency.

If the node divides the length of shaft into segments l_1 and l_2, we have

$$\alpha = \sqrt{\frac{CI_0}{M_1k_1{}^2l_1}} = \sqrt{\frac{CI_0}{M_2k_2{}^2l_2}}.$$

Hence
$$\frac{l_1}{l_2} = \frac{M_2k_2{}^2}{M_1k_1{}^2},$$

showing that the shaft is divided inversely as the moments of inertia of the masses.

Then
$$\alpha = \sqrt{\frac{CI_0}{l}\left(\frac{M_1k_1{}^2 + M_2k_2{}^2}{M_1k_1{}^2 . M_2k_2{}^2}\right)}.$$

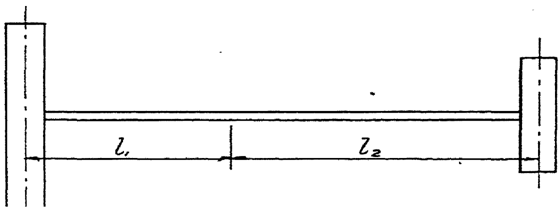


FIG. 30.

Collecting our results and substituting c for $\frac{CI_0}{l}$ and N for Mk^2 to make the expressions less cumbersome, we have:

For a single mass carried on a shaft fixed at the other end,

$$\alpha = \sqrt{\frac{CI_0}{Mk^2l}} = \sqrt{\frac{c}{N}}.$$

For a single mass carried on a composite shaft fixed at both ends,

$$\alpha = \sqrt{\frac{C}{N}\left(\frac{I_1}{l_1} + \frac{I_2}{l_2}\right)} = \sqrt{\frac{c_1}{N} + \frac{c_2}{N}}.$$

For two masses on a free-ended shaft,

$$\alpha = \sqrt{\frac{CI_0}{l}\left(\frac{M_1k_1{}^2 + M_2k_2{}^2}{M_1k_1{}^2 . M_2k_2{}^2}\right)} = \sqrt{c\left(\frac{1}{N_1} + \frac{1}{N_2}\right)}.$$

These expressions will enable us to obtain a solution for the next case.

Three Masses carried on a Free-ended Composite Shaft.

Suppose three masses to be arranged as in Fig. 31. The system will be capable of oscillating in two ways, for oscillations may be set up either by twisting the two outer masses in opposite directions, or by twisting the outer masses in the same direction while the middle mass is twisted in the opposite direction.

Let the masses have moments of inertia N_1, N_2 and N_3, the shaft portions lengths l_1 and l_3, and the torque necessary to twist the shaft portions through unit angle be c_1 and c_3.

For a system consisting of the middle mass and shaft portion l_1 fixed at the end we may write

$$\alpha_1^2 = \frac{c_1}{N_2}.$$

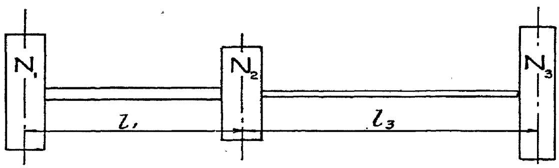


FIG. 31.

For a system consisting of the middle mass and shaft portion l_3, we may write

$$\alpha_2^2 = \frac{c_3}{N_2}.$$

For a system consisting of the first end mass and the middle mass on shaft portion l_1, the ends being regarded as free, we may write

$$\alpha_3^2 = \frac{c_1}{N_1} + \frac{c_1}{N_2},$$

and for a similar system consisting of the other end mass and the middle mass on shaft portion l_3, we have

$$\alpha_4^2 = \frac{c_3}{N_2} + \frac{c_3}{N_3}.$$

In the actual vibration, nodes may fall in each of the portions l_1 and l_3. Let them be at distances x and y respectively from the ends. The arrangement may be resolved

into three separate systems which will each have the same frequency, one consisting of the mass of inertia N_1, mounted on a shaft of length x fixed at the end, the next consisting of the mass of inertia N_2, mounted on a fixed composite shaft, the lengths of the parts of which are $l_1 - x$ and $l_2 - y$, and, finally, one consisting of the mass of inertia N_3, mounted on a shaft of length y fixed at the end. We have

$$\alpha^2 = \frac{c_1 l_1}{N_1 x} = \frac{c_1 l_1}{N_2(l_1 - x)} + \frac{c_3 l_3}{N_2(l_3 - y)} = \frac{c_3 l_3}{N_3 y}.$$

Eliminating $\frac{x}{l_1}$, and $\frac{y}{l_3}$, and reducing, we get

$$(\alpha^2 - \alpha_3^2)(\alpha^2 - \alpha_4^2) = \alpha_1^2\alpha_2^2$$

and $$\alpha^2 = \tfrac{1}{2}(\alpha_3^2 - \alpha_4^2) \pm \tfrac{1}{2}\{(\alpha_3^2 - \alpha_4^2)^2 + 4\alpha_1^2\alpha_2^2\}^{\frac{1}{2}}.$$

The two roots give the solutions for the two modes of vibration indicated.

Numerical Example of Three-Mass System in Torsional Vibration.

Let $N_1 = 1{,}000$ pounds (feet)^2,

$N_2 = 20{,}000$ " "

$N_3 = 6{,}000$ " "

$c_1 = 120 \times 10^6$ foot-poundals.

$c_3 = 24 \times 10^6$ " "

Then $$\alpha_1^2 = \frac{c_1}{N_2} = \cdot 6 \times 10^4,$$

$$\alpha_2^2 = \frac{c_3}{N_2} = \cdot 12 \times 10^4,$$

$$\alpha_3^2 = \frac{c_1}{N_1} + \frac{c_1}{N_2} = 12 \times 10^4 + \cdot 6 \times 10^4 = 12\cdot 6 \times 10^4,$$

$$\alpha_4^2 = \frac{c_3}{N_3} + \frac{c_3}{N_2} = \cdot 4 \times 10^4 + \cdot 12 \times 10^4 = \cdot 52 \times 10^4,$$

$$\begin{aligned}\alpha^2 &= \tfrac{1}{2}(\alpha_3^2 + \alpha_4^2) \pm \tfrac{1}{2}\{(\alpha_3^2 - \alpha_4^2)^2 + 4\alpha_1^2\alpha_2^2\}^{\frac{1}{2}} \\ &= \tfrac{1}{2}(13\cdot 12 \times 10^4) \pm \tfrac{1}{2}\{(12\cdot 08 \times 10^4)^2 + \cdot 288 \times 10^8\}^{\frac{1}{2}} \\ &= 6\cdot 56 \times 10^4 \pm 6\cdot 05 \times 10^4 \\ &= 12\cdot 61 \times 10^4 \text{ and } \cdot 51 \times 10^4,\end{aligned}$$

$$\alpha = 355 \text{ and } 71\cdot 4.$$

and $$T = \frac{2\pi}{355} \text{ and } \frac{2\pi}{71\cdot 4} = \cdot 0177 \text{ and } \cdot 088 \text{ second.}$$

Transverse Vibrations of Beams and Whirling of Shafts

Bending of Elastic Beams.

In the case of a vibrating beam, the oscillating masses produce a bending moment, which is resisted by a controlling couple called up in the beam, due to the elasticity of the material.

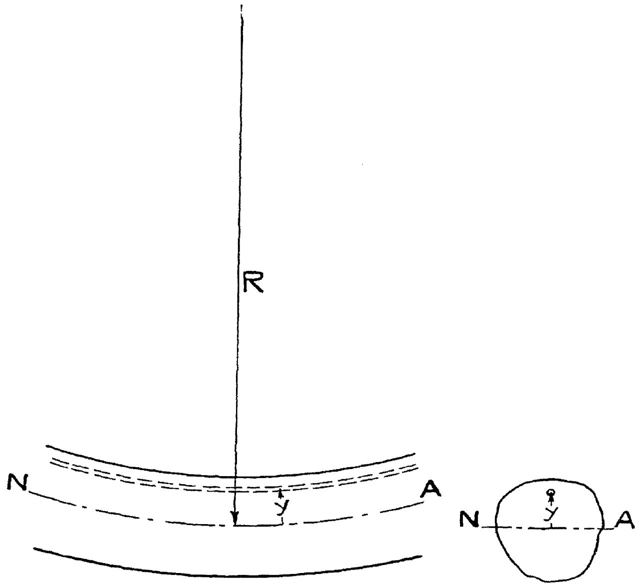


Fig. 32.

Figure 32 shows the portion of a beam subjected to a bending moment of magnitude M. The top fibres are in compression and the bottom fibres in tension. Fibres on an intermediate plane will be unstrained. This plane is known as the *Neutral Axis.*

Let R be the radius of curvature. Then the length of unstrained neutral fibre is $R\theta$, that of a strained fibre at

distance y from the neutral fibre is $(R \pm y)\,\theta$, and the strain

$$e = \frac{(R \pm y)\,\theta - R\theta}{R\theta} = \frac{\pm y}{R}.$$

$$E = \frac{f}{e},\ i.e.\ f = eE = \pm y\frac{E}{R}.$$

The external bending moment is resisted by an equal and opposite internal couple of magnitude M.

The moment represented by the forces acting on a small area a at a distance y from the neutral axis is fay. The total moment is the sum of all such moments, so that

$$M = \Sigma fay, \text{ but } f = \frac{Ey}{R};$$

therefore $$M = \Sigma \frac{E}{R}ay^2 = \frac{E}{R}\Sigma ay^2.$$

But Σay^2 is the second moment of area for the section and is equal to I, so that $$M = \frac{EI}{R}.$$

It will be convenient to express R in terms of Cartesian co-ordinates. It may be shown that

$$\frac{1}{R} = \frac{\dfrac{d^2y}{dx^2}}{\left\{1 + \left(\dfrac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}},$$

which, if the curvature is very small, may be taken as $\dfrac{d^2y}{dx^2}$

which gives $$M = EI\frac{d^2y}{dx^2},$$

x being the horizontal distance of the section from a convenient point of reference, and y being the deflection.

Relation between Intensity of Loading, Shearing Force, Bending Moment, Slope and Deflection of a Variably Loaded Beam.

The usual symbols are w = intensity of loading in poundals per foot run; S = shearing force in poundals; M = bending

moment in foot-poundals ; i = slope in radians ; y = deflection in feet.

If w can be expressed as some function of the horizontal distance x from a point of reference, say a point of support, it may be shown that

$$S = \int w dx,$$

$$M = \int S dx,$$

$$i = \frac{1}{EI}\int M dx,$$

and $$y = \frac{1}{EI}\int i dx.$$

Hence $$\frac{d^4y}{dx^4} = \frac{w}{EI}.$$

General Case of the Whirling of an Unloaded Shaft.

Let the mass of the shaft be m pounds per foot run, let it be

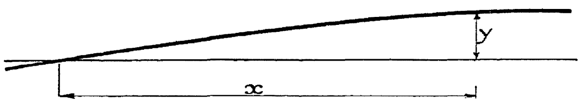


Fig. 33.

rotated with an angular velocity ω, and let the deflection at any point distant x from some point of reference be y.

The centrifugal force on an element of unit length will be $m\omega^2 y$ poundals, where $m\omega^2 y$ will be equivalent to w, the intensity of loading in what goes before.

Then $$\frac{d^4y}{dx^4} = \frac{m\omega^2}{EI}y.$$

Let $$\frac{m\omega^2}{EI} = u^4. \quad \text{Then } \frac{d^4y}{dx^4} = u^4 y.$$

The general solution of this is :

$$y = A \cosh ux + B \sinh ux + C \cos ux + D \sin ux \quad (1),$$

from which

$$\frac{dy}{dx} = u(\text{A} \sinh ux + \text{B} \cosh ux - \text{C} \sin ux + \text{D} \cos ux \quad (2),$$

$$\frac{d^2y}{dx^2} = u^2(\text{A} \cosh ux + \text{B} \sinh ux - \text{C} \cos ux - \text{D} \sin ux \quad (3),$$

and $\frac{d^3y}{dx^3} = u^3(\text{A} \sinh ux + \text{B} \cosh ux + \text{C} \sin ux - \text{D} \cos ux \quad (4)$

For special cases, the values of the constants A, B, C and D may be determined from knowledge of the values of deflection, slope, bending moment and shear for various special values of x.

SPECIAL CASES.

(1) *Uniform Shaft freely supported at ends.*

When $x = \text{O}$, $y = \text{O}$ and $\frac{d^2y}{dx^2} = \text{O}$.

Substituting $x = \text{O}$ in (1) and (3) we have

$$\text{O} = \text{A} + \text{C} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (a)$$

and

$$\text{O} = \text{A} - \text{C} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (b)$$

Also when $x = \text{L}$, $y = \text{O}$ and $\frac{d^2y}{dx^2} = \text{O}$.

Substituting $x = \text{L}$ in (1) and (3) we have

$$\text{O} = \text{A} \cosh u\text{L} + \text{B} \sinh u\text{L} + \text{C} \cos u\text{L} + \text{D} \sin u\text{L}, \quad (c)$$

and $\text{O} = \text{A} \cosh u\text{L} + \text{B} \sinh u\text{L} - \text{C} \cos u\text{L} - \text{D} \sin u\text{L} \quad (d)$

From (a) and (b) $\text{A} = \text{C} = \text{O}$. Hence, rewriting (c) and (d),

$$\text{O} = \text{B} \sinh u\text{L} + \text{D} \sin u\text{L}$$

and

$$\text{O} = \text{B} \sinh u\text{L} - \text{D} \sin u\text{L},$$

from which $\text{O} = \text{D} \sin u\text{L}$. But if $\text{D} = \text{O}$ the shaft will always have zero deflection, which is not the case, so we must have $\sin u\text{L} = \text{O}$. Therefore $u\text{L} = \pi$ or 2π or 3π or 4π, etc., corresponding to the fundamental and harmonic modes of vibration.

Taking the fundamental, or slowest frequency,

$$u\mathrm{L} = \pi\,;\ u^4\mathrm{L}^4 = \pi^4\,;\ \frac{m\omega^2}{\mathrm{EI}}\mathrm{L}^4 = \pi^4\,;$$

$$\omega = \frac{\pi^2}{\mathrm{L}^2}\sqrt{\frac{\mathrm{EI}}{m}}.$$

The above will give correct results if the foot is used consistently as the unit of length, *i.e.* if L be expressed in feet; E in poundals per square foot; I in (feet)^4 and m in pounds per foot run.

More generally,

$$u\mathrm{L} = k\pi \text{ where } k = 1,\ 2,\ 3,\ 4,\ 5, \text{ etc.}$$

and
$$\omega = k^2\frac{\pi^2}{\mathrm{L}^2}\sqrt{\frac{\mathrm{EI}}{m}}.$$

It is thus seen that ω varies as k^2, *i.e.* the frequencies vary as 1, 4, 9, 16, 25, etc.

(2) *Uniform Shaft fixed in direction at one end and free at the other.*

In this case when $x = \mathrm{O}$, $y = \mathrm{O}$ and $\frac{dy}{dx} = \mathrm{O}$, and when

$$x = \mathrm{L},\ \frac{d^2y}{dx^2} = \mathrm{O} \text{ and } \frac{d^3y}{dx^3} = \mathrm{O}.$$

Substituting $x = \mathrm{O}$ in (1), $x = \mathrm{O}$ in (2); $x = \mathrm{L}$ in (3) and $x = \mathrm{L}$ in (4)

we find that
$$\cosh u\mathrm{L} \cos u\mathrm{L} = -1,$$

or
$$\cosh u\mathrm{L} = \quad -1$$

If the values of $\cosh x$ and $\frac{-1}{\cos x}$ are plotted against x, it will be found that the curves intersect at $x = 1{\cdot}875$. The solution therefore is $u = \frac{1{\cdot}875}{\mathrm{L}}$.

But
$$u^4 = \frac{m\omega^2}{\mathrm{EI}},$$

so that $$\omega = u^2\sqrt{\frac{\mathrm{EI}}{m}} = \frac{3 \cdot 52}{\mathrm{L}^2}\sqrt{\frac{\mathrm{EI}}{m}}.$$

(3) *Uniform Shaft fixed in direction at both ends.*

In this case when $x = \mathrm{O}$, $y = \mathrm{O}$ and $\frac{dy}{dx} = \mathrm{O}$, also when

$$x = \mathrm{L},\ y = \mathrm{O} \text{ and } \frac{dy}{dx} = \mathrm{O}.$$

Substituting $x = \mathrm{O}$ in (1); $x = \mathrm{O}$ in (2); $x = \mathrm{L}$ in (1); and $x = \mathrm{L}$ in (2),

we find that $\cos u\mathrm{L} \cosh u\mathrm{L} = 1$; or $\cos u\mathrm{L} = \frac{1}{\cosh x}$.

Plotting values of $\cos x$ and $\frac{}{\cosh x}$

against x, the value of x for the points where the curves intersect gives a critical value for $u\mathrm{L}$ of 4·73.

Hence $$\omega = u^2\sqrt{\frac{\mathrm{EI}}{m}} = \frac{22 \cdot 3}{\mathrm{L}^2}\sqrt{\frac{\mathrm{EI}}{m}}$$

(4) *Uniform Shaft fixed in direction at one end and freely supported at the other.*

In this case when $x = \mathrm{O}$, $y = \mathrm{O}$ and $\frac{dy}{dx} = \mathrm{O}$, also when

$$x = \mathrm{L},\ y = \mathrm{O} \text{ and } \frac{d^2y}{dx^2} = \mathrm{O}.$$

Substituting $x = \mathrm{O}$ in (1); $x = \mathrm{O}$ in (2); $x = \mathrm{L}$ in (1) and $x = \mathrm{L}$ in (3),

we find that $$\cot u\mathrm{L} = \coth u\mathrm{L}.$$

Plotting values of $\cot x$ and $\coth x$, the value of x for the point where the curves intersect will give a critical value of $u\mathrm{L}$ of 3·93.

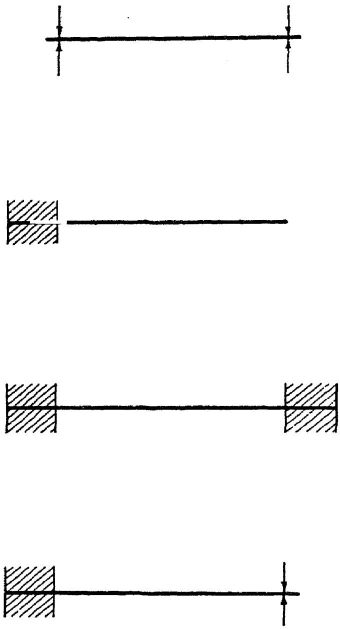
Hence $$\omega = u^2\sqrt{\frac{\mathrm{EI}}{m}} = \frac{15 \cdot 45}{\mathrm{L}^2}\sqrt{\frac{\mathrm{EI}}{m}}.$$

For all the cases considered there will be higher critical speeds, but, in most practical cases, only the first is of importance. This is especially so when the shaft carries concentrated loads in addition.

For the purpose of comparison, it may be noted that the solutions for the first critical speed are of the form

$$\omega = \frac{\ }{L^2}\sqrt{\frac{EI}{m}}$$

and the values are tabulated in Fig. 34.



$$\omega = \frac{\pi^2}{L^2}\sqrt{\frac{EI}{m}}.$$

$$\omega = \frac{352}{L^2}\quad\frac{EI}{m}.$$

$$\omega = \frac{22 \cdot 3}{L^2}\sqrt{\frac{EI}{m}}.$$

$$\omega = \frac{15 \cdot 45}{L^2}\sqrt{\frac{EI}{m}}.$$

FIG. 34.

Transverse Vibration of a Uniform Bar.

If m is the mass in pounds per foot run, and y is the deflection in feet at a point distant x feet from a convenient point of reference, then at any instant in the course of a vibration the intensity of loading is equal and opposite to $m\ddot{y}$. The deflection of the beam at any point due to the application of a varying load, either at that point or at any other point, is proportional to the magnitude of the applied load. Thus for

every point in the beam $m\ddot{y} = -fy$, where f is a constant. Hence $\ddot{y} = -\frac{f}{m}y$, which is of simple harmonic form and may be written $\ddot{y} = -\alpha^2 y$, where $\alpha = \sqrt{\frac{f}{m}}$, a constant.

Since the intensity of loading is equal and opposite to $m\ddot{y}$, it is equal to $m\alpha^2 y$, and we have, as in the case of whirling, $\frac{d^4y}{dx^4} = \frac{m\alpha^2 y}{\mathrm{EI}}$. The general solution and special solutions for various conditions of end support will also be identical.

Transverse Vibrations and Whirling of Uniform Shafts carrying Concentrated Masses.

For the present it will be assumed that the mass of the shaft carrying the masses is negligible compared with the masses themselves.

From the theory of the deflection of beams it will be shown that the deflection due to a force, measured at the point of application of the force, is of the form

$$y = a\frac{\mathrm{PL}^3}{\mathrm{EI}},$$

where P = force in poundals, L = length of shaft in feet, E = coefficient of elasticity in poundals per square foot, I = second moment of area of section in (feet)^4, and a is a coefficient dependent on the position of the mass and the method of fixing at the ends of the span.

It is important to note that, for a given shaft fixed in any one manner, the deflection under the load is simply proportional to the force P producing the deflection.

To simplify the expression, write $y = a \, . \, b \, . \, \mathrm{P}$, where

$$b = \frac{\mathrm{L}^3}{\mathrm{EI}}.$$

Then $\mathrm{P} = \frac{y}{ab}$: that is to say, if the shaft is displaced through a distance y, the force tending to restore the shaft to its position of rest is $\frac{y}{ab}$, but in the case of a shaft in vibration, at any instant this is equal and opposite to $\mathrm{M}\ddot{y}$.

We may therefore write

$$M\ddot{y} = -\frac{y}{ab}; \text{ or } \ddot{y} = -\frac{y}{abM}.$$

This is of the form $\ddot{y} = -\alpha^2 y$, where $\alpha = \sqrt{\frac{1}{abM}}$.

Therefore $$T = \frac{2\pi}{\alpha} = 2\pi\sqrt{abM}.$$

Now let the shaft be fixed horizontally. It will then deflect a distance y' under a force P produced by the weight of the mass M, and equal to Mg poundals, and we shall have

$$P = \frac{y}{ab}, \ Mg = \frac{y'}{ab}, \text{ and } abM = \frac{y'}{g}.$$

Substituting in the previous expressions for α and T, we have

$$\alpha^2 = \frac{g}{y'}, \text{ and } T = 2\pi\sqrt{\frac{y'}{g}},$$

which is a very simple but important result, and gives the period of transverse vibration for any shaft whatever, carrying a mass M, if y' is the deflection in the plane of the mass due to its weight. The mass of the shaft is assumed negligible compared with the load. It will be noticed that the form of the expression is the same as that for the time of oscillation of a simple pendulum, the value of the deflection y' taking the place of the length of the pendulum.

Now consider the analogous case of a whirling mass. The force P producing the deflection is then due to the centrifugal force of the mass M instead of to its weight. We have $P = \frac{y}{ab}$, or

$$M\omega^2 y = \frac{y}{ab}, \text{ or } M\omega^2 = \frac{1}{ab}.$$

But if y' be the deflection produced by the weight of the mass M

$$Mg = \frac{y'}{ab}.$$

Combining these results, we find that $\omega^2 = \frac{1}{abM} = \frac{g}{y^1}$. Again

a most important result, which gives the critical speed for any weightless shaft carrying a single concentrated mass simply in terms of its deflection under gravity. For $\omega =$ angular velocity at critical speed, in radians per second, $g =$ acceleration due to gravity in feet per (second)^2, $y' =$ deflection under gravity of the shaft in feet measured or calculated for the plane of the mass.

In what follows immediately it is shown how to calculate the deflection under a single concentrated mass with various conditions of end fixing.

(1) *Uniform Shaft freely supported at ends and carrying a Concentrated Mass at any Point.*

Let M be the mass in pounds and R_A and R_B be the reactions at the supports, then

$$R_A = Mg\frac{b}{L} \text{ and } R_B = Mg\frac{a}{L}.$$

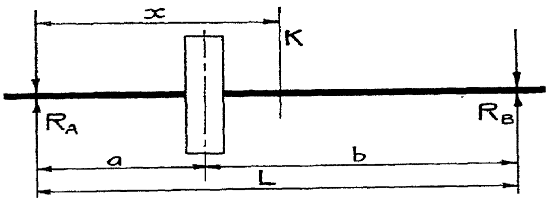


FIG. 35.

Then the bending moment at any section K distant x from A is given by the equation

$$EI\frac{d^2y}{dx^2} = R_Ax - Mg(x - a).$$

Integrating for the slope:

$$EI\frac{dy}{dx} = \frac{R_Ax^2}{2} - Mg\frac{(x-a)^2}{2} + C.$$

Integrating again for deflection:

$$EIy = \frac{R_Ax^3}{6} - Mg\frac{(x-a)^3}{6} + Cx + D.$$

When $x = O$, $y = O$ and $D = O$, and when $x = L$, $y = O$, and

$$C = \frac{Mgb}{6L}(b^2 - L^2).$$

For deflection under load, put $x = a$.

Then $$EIy = Mg \cdot \frac{b}{L} \cdot \frac{a^3}{6} + Mg\frac{ab}{6L}(b^2 - L^2)$$

$$= Mg\frac{ab}{6L}(a^2 + b^2 - L^2)$$

$$= Mg\frac{ab}{6L}(a^2 + b^2 - a^2 - 2ab - b^2)$$

$$= - Mg\frac{}{3L},$$

so that $$y = -\frac{Mg}{EI} \cdot \frac{a^2b^2}{3L}.$$

In using this and similar formulæ, M is expressed in pounds; E in poundals per square foot; I in $(\text{feet})^4$, a, b, L and y in feet.

(2) *Uniform Shaft fixed in direction at one end and free at the other.*

Let $M =$ mass in pounds; R_A reaction at support $= Mg$; N_A bending moment at support $= Mga$; then the bending

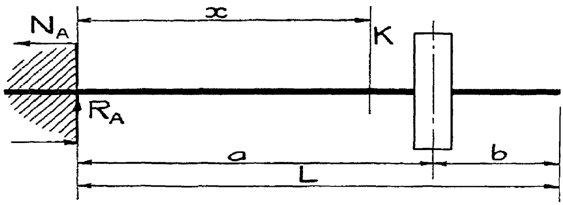


FIG. 36.

moment at any section K distant x from A is given by the equation:

$$EI\frac{d^2y}{dx^2} = R_A x - Mga.$$

Integrating for slope :

$$EI\frac{dy}{dx} = \frac{R_A x^2}{2} - Mgax + C.$$

Integrating again for deflection :

$$EIy = \frac{R_A x^3}{6} - Mg\frac{ax^2}{2} + Cx + D.$$

When $x = O$, $y = O$ and $D = O$; also

$$\frac{dy}{dx} = O \text{ and } C = O.$$

For deflection under the load put $x = a$

$$EIy = Mg\frac{a^3}{6} - Mg\frac{a^3}{2} = -Mg\frac{a^3}{3},$$

so that
$$y = -\frac{Mg}{EI}\cdot\frac{a^3}{3}.$$

(3) *Uniform Shaft fixed in direction at both ends.*

The bending moment at any section K distant x from A is given by the equation

$$EI\frac{d^2y}{dx^2} = R_A x - N_A - Mg(x - a).$$

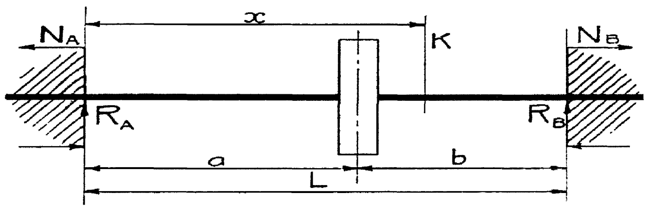


FIG. 37.

Integrating for slope :

$$EI\frac{dy}{dx} = R_A\frac{x^2}{2} - N_A x - Mg\frac{(x-a)^2}{2} + C.$$

Integrating again for deflection :

$$EIy = R_A\frac{x^3}{6} - N_A\frac{x^2}{2} - Mg\frac{(x-a)^3}{6} + Cx + D.$$

When $x = O$, $y = O$, $D = O$, $\frac{dy}{dx} = O$ and $C = O$.

To find R_A and N_A.

When $x = L$, $\frac{dy}{dx} = O$ and $y = O$, so that we have

$$R_A\frac{L^2}{2} - N_A L - Mg\frac{b^2}{2} = O,$$

$$R_A\frac{L^3}{6} - N_A\frac{L^2}{2} - Mg\frac{b^3}{6} = O.$$

Hence
$$R_A = Mg\frac{b^2}{L^3}(b + 3a),$$

and
$$N_A = \frac{Mg}{2L^2}b^2(b + 3a) - \frac{Mg}{2L}b^2.$$

Substituting these values and putting $x = a$ in the expression for the deflection, we get for the deflection under the load

$$EIy = -Mg\frac{a^3b^3}{3L^3},$$

so that
$$y = -\frac{Mg}{EI}\cdot\frac{a^3b^3}{3L^3}.$$

(4) *Uniform Shaft fixed in direction at one end and freely supported at the other.*

The bending moment at any section K distant x from A is given by the equation

$$EI\frac{d^2y}{dx^2} = R_A x - N_A - Mg(x - a).$$

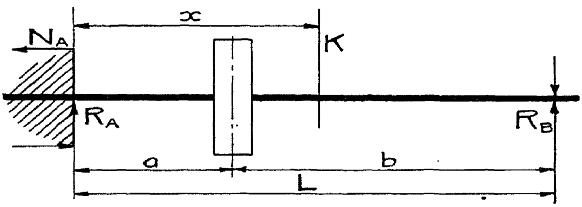


FIG. 38.

Integrating for the slope:

$$EI\frac{dy}{dx} = R_A\frac{x^2}{2} - N_A x - Mg\frac{(x-a)^2}{2} + C.$$

Integrating for the deflection:

$$EIy = R_A\frac{x^3}{6} - N_A\frac{x^2}{2} - Mg\frac{(x-a)^3}{6} + Cx + D.$$

When $x = O$, $y = O$, $D = O$, $\frac{dy}{dx} = O$ and $C = O$.

To find R_A and N_A

when $\qquad x = L,\ \frac{d^2y}{dx^2} = O$ and $y = O$.

Therefore $\qquad R_A L - N_A - Mgb = O,$

$$R_A\frac{L^3}{6} - N_A\frac{L^2}{2} - Mg\frac{b^3}{6} = O.$$

Hence $\qquad R_A = Mg\frac{b}{2L^3}\cdot(3L^2 - b^2),$

and $\qquad N_A = Mg\frac{b}{2L^2}\cdot(L^2 - b^2).$

Substituting these values, and putting $x = a$ in the expression for the deflection, then the deflection under load is given by

$$y = -\frac{Mg}{EI}\frac{a^3b^2}{12L^3}(3a + 4b).$$

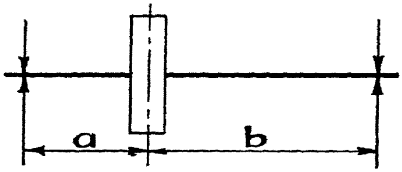
Collecting our results for simple concentrated loads for the various conditions of end supports we have the values shown in Fig. 39, and in each case the critical speed is given by

$$\omega = \sqrt{\frac{g}{y}}.$$

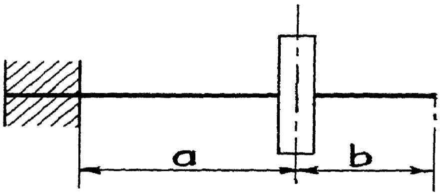
Empirical Formula for Complex Systems.

It has been found experimentally that if ω_1, ω_2, ω_3 are the critical angular velocities of separate vibrating or whirling systems, that of a complex system made up of a combination of the elements of the separate systems will be given by

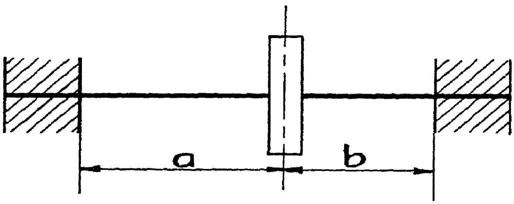
$$\frac{1}{\omega^2} = \frac{1}{\omega_1^2} + \frac{1}{\omega_2^2} + \frac{1}{\omega_3^2} + \text{, etc.}$$



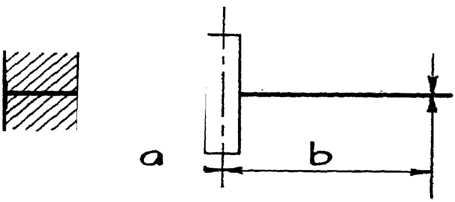
$$y = \frac{Mg}{EI}\,\frac{a^2b^2}{3L}.$$



$$y = \frac{Mg}{EI}\,\frac{a^3}{3}.$$



$$y = \frac{Mg}{EI}\,\frac{a^3b^3}{3L^3}.$$



$$y = \frac{Mg}{EI}\,\frac{a^3b^2}{12L^3}(3a + 4b).$$

FIG. 39.

Thus if ω_1, ω_2, ω_3, etc., represent the angular velocities at the critical speed of systems carrying concentrated masses M_1, M_2, M_3, etc., in each case on a shaft which may be assumed to be without mass, but having a definite method of end fixing, we have

$$\omega_1^2 = \frac{g}{y_1}, \quad \omega_2^2 = \frac{g}{y_2}, \quad \omega_3^2 = \frac{g}{y_3}, \text{ etc.}$$

Therefore, $$\frac{1}{\omega^2} = \frac{1}{g}\Sigma y.$$

If it is desired to allow for the mass of the shaft it may be treated as an element in the complex system.

Let $\omega_0 =$ the critical angular velocity of the unloaded shaft with the defined method of end support, then the critical angular velocity of the same shaft carrying a number of concentrated loads is given by

$$\frac{1}{\omega^2} = \frac{1}{\omega_0{}^2} + \frac{1}{g}\Sigma y.$$

Methods have been given of calculating ω_0 and also values of y for any usual method of end support.

To show how they may be applied, we will consider a practical case of a turbine shaft carrying concentrated loads and supported in three bearings.

In Fig. 40, the loads carried on a turbine shaft together with the position of their points of application are shown. The mean diameter of the shaft is 0·5 foot.

It will be necessary to exercise some judgment as to the conditions of end support which apply in the particular case.

It is open to us to calculate the critical speed of the main span regarding the shaft as freely supported at bearings Q and R.

For these conditions, the equations applicable will be:—
For the critical angular velocity of the unloaded shaft

$$\omega_0 = \frac{\pi^2}{L^2}\sqrt{\frac{EI}{m}},$$

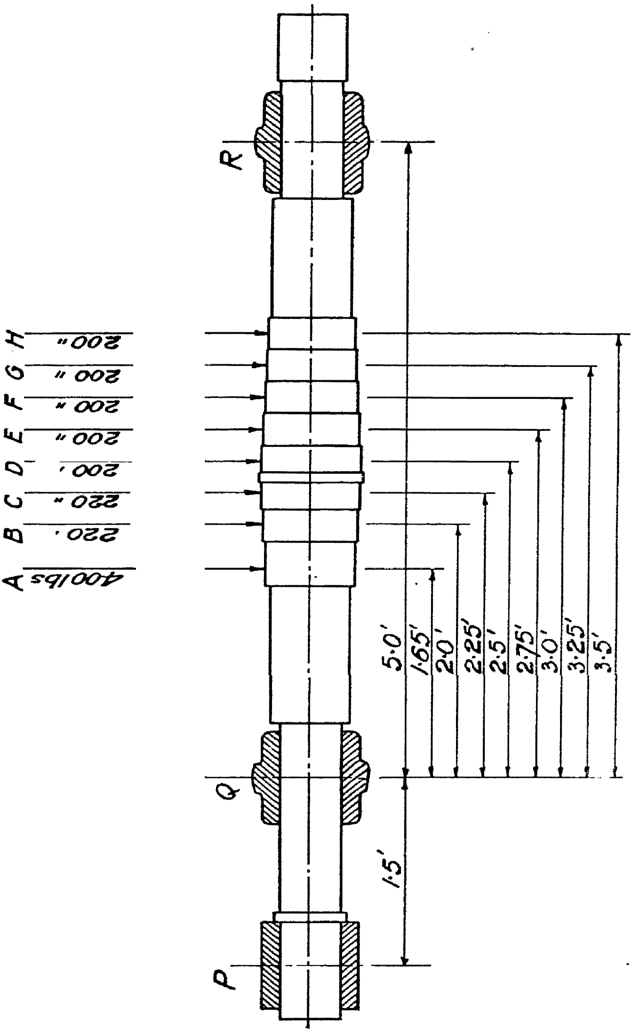

A
B
C
D
E
F
G
H
400lbs
220
220
200
200
200
200
200
R
Q
P
5·0'
1·65'
2·0'
2·25'
2·5'
2·75'
3·0'
3·25'
3·5'
1·5'

Fig. 40.

and for the deflection due to the separate application of each load, under the load in question:

$$y = \frac{Mg}{EI} \cdot \frac{a^2b^2}{3L}.$$

This method is worked out as a matter of interest, but will not give the true critical speed owing to the presence of the third bearing P, which will have the effect of making the conditions of support at Q approximate very closely to that of a shaft fixed in direction.

The most nearly correct result will be obtained by making calculations for the main span on the assumption that the shaft is fixed in direction at Q and freely supported at R.

The equations most nearly applicable will then be:—For the critical angular velocity of the unloaded shaft

$$\omega_c = \frac{15{\cdot}45}{L^2}\sqrt{\frac{EI}{m}},$$

and for the deflection due to the separate application of each load under the load in question

$$y = \frac{Mg}{EI} \cdot \frac{a^3b^2}{12L^3}(3a + 4b).$$

Again, as a matter of interest, the bearings Q and R might both be regarded as fixing the direction of the shaft, in which case the equations to be used are:—

For the critical angular velocity of the unloaded shaft

$$\omega_c = \frac{22{\cdot}3}{L^2}\sqrt{\frac{EI}{m}},$$

and for the deflection due to the separate application of each load under the load in question

$$y = \frac{Mg}{EI} \cdot \frac{a^3b^3}{3L^3}.$$

All three cases are worked out in what follows.

$E = 30{,}000{,}000\ g \times 144$ poundals per square foot.

$I = \frac{\pi}{4} \cdot 25^4 = 00308$ feet^4.

$m = 95$ pounds per foot length.

$L = 5$ feet.

Case I.—Assumed freely supported at Q and R.

$$\omega_0 = \frac{\pi^2}{L^2}\sqrt{\frac{EI}{m}}$$

$$= \frac{\pi^2}{25}\sqrt{\frac{30{,}000{,}000\, g \times 144 \times \cdot 00308}{95}}$$

$$= 841 \text{ radians per second}$$

$$= 8{,}040 \text{ revolutions per minute,}$$

giving the critical speed of the unloaded shaft when freely supported.

For the deflection due to each load

$$y = \frac{Mg}{EI}\cdot\frac{a^2b^2}{3L} = \frac{Ma^2b^2}{199{,}600{,}000}.$$

	M	a	b	Ma^2b^2
A	400	1.65	3·35	12,250
B	220	2·0	3·0	7,920
C	220	2·25	2·75	8,450
D	200	2·5	2·5	7,800
E	200	2·75	2·25	7,670
F	200	3·0	2·0	7,200
G	200	3·25	1·75	6,480
H	200	3·5	1·5	5,520
				63,290

Then for the critical angular velocity of the combination

$$\frac{1}{\omega^2} = \frac{1}{\omega_0{}^2} + \frac{1}{g}\Sigma y$$

$$= \frac{1}{841^2} + \frac{1}{g}\cdot\frac{\Sigma Ma^2b^2}{199{,}600{,}000}$$

$$= \frac{1}{841^2} + \frac{63{,}290}{199{,}600{,}000\, g} = \frac{1}{88{,}800}.$$

Therefore $\omega = 298$ radians per second
$= 2{,}850$ revolutions per minute.

Case II.—Assumed fixed in direction at Q and freely supported at R.

$$\omega_1 = \frac{15\cdot45}{L^2}\sqrt{\frac{EI}{m}}$$

$$= \frac{15\cdot45}{25}\sqrt{\frac{30{,}000{,}000\, g \times 144 \times \cdot00308}{95}}$$

$= 1{,}312$ radians per second
$= 12{,}530$ revolutions per minute,

giving the critical speed of the unloaded shaft for this condition of end support.

For the deflection under each load,

$$y = \frac{Mg}{EI}\cdot\frac{a^3b^2}{12L^3}(3a+4b)$$

$$= \frac{Ma^3b^2(3a+4b)}{199{,}600{,}000 \times 100}.$$

	M	a	b	$Ma^3b^2(3a+4b)$
A	400	1·65	3·35	371,000
B	220	2·0	3·0	285,000
C	220	2·25	2·75	337,000
D	200	2·5	2·5	341,000
E	200	2·75	2·25	364,000
F	200	3·0	2·0	368,000
G	200	3·25	1·75	353,000
H	200	3·5	1·5	319,000
				2,738,000

For the critical angular velocity of the combination,

$$\frac{1}{\omega^2} = \frac{1}{\omega_0{}^2} + \frac{1}{g}\Sigma y$$

$$= \frac{1}{1{,}312^2} + \frac{1}{g} \times \frac{\Sigma \mathrm{M}a^3b^2(3a + 4b)}{199{,}600{,}000 \times 100}$$

$$= \frac{1}{1{,}312^2} + \frac{1}{g} \times \frac{2{,}738{,}000}{199{,}600{,}000 \times 100}$$

$$= \frac{1}{207{,}000}.$$

Therefore $\omega = 456$ radians per second
$= 4{,}360$ revolutions per minute.

Case III.—Assumed fixed in direction at both Q and R.

$$\omega_0 = \frac{22 \cdot 3}{\mathrm{L}^2} \cdot \sqrt{\frac{\mathrm{EI}}{m}}$$

$$= \frac{22 \cdot 3}{25}\sqrt{\frac{30{,}000{,}000\, g \times 144 \times \cdot 00308}{95}}$$

$= 1{,}900$ radians per second
$= 18{,}140$ revolutions per minute,

giving the critical speed of the unloaded shaft with the ends fixed in direction.

For the deflection under each load,

$$y = \frac{\mathrm{M}g}{\mathrm{EI}} \cdot \frac{a^3b^3}{3\mathrm{L}^3}$$

$$\frac{\mathrm{M}a^3b^3}{199{,}600{,}000 \times 25}.$$

	M	a	b	$\mathrm{M}a^3b^3$
A	400	1·65	3·35	67,750
B	220	2·0	3·0	47,520
C	220	2·25	2·75	52,300

	M	a	b	Ma^3b^3
D	200	2·5	2·5	48,700
E	200	2·75	2·25	47,500
F	200	3·0	2·0	43,200
G	200	3·25	1·75	36,900
H	200	3·5	1·5	29,000
				372,870

For the critical angular velocity of the combination,

$$\frac{1}{\omega^2} = \frac{1}{\omega_0} + \frac{1}{g}\Sigma y$$

$$= \frac{1}{1,900^2} + \frac{1}{g}\,\frac{\Sigma M a^3 b^3}{199,600,000 \times 25}$$

$$= \frac{1}{1,900^2} + \frac{1}{g}\cdot\frac{372,870}{199,600,000 \times 25}$$

$$= \frac{1}{386,000}.$$

Therefore $\omega = 622$ radians per second

$= 5,940$ revolutions per minute.

As already stated, the conditions of the practical case are very closely represented by those of Case II in which the shaft is assumed fixed in direction at Q and freely supported at R. For this case, the critical speed is equal to 4,360 r.p.m.

Determination of Angular Deviation from the Mean Position in the Case of an Engine and Flywheel working against a constant External Load Torque.

Let $T =$ torque exerted by the engine at any instant, $T_m =$ constant external load torque and also mean engine torque, Mk^2 moment of inertia of flywheel, $\theta =$ angular deviation from the mean position, $\omega =$ angular velocity at any instant, $\omega_m =$ mean angular velocity.

At any instant, the difference between the engine torque and the constant external resisting torque will be equal to the rate of change of angular momentum of the flywheel, so that $(T - T_m) = Mk^2\ddot{\theta}$ and

$$\ddot{\theta} = \frac{1}{Mk^2}(T - T_m).$$

This quantity integrated with respect to time will give the change from the mean angular velocity, *i.e.*

$$\dot{\theta} = \frac{1}{Mk^2}\int(T - T_m)\,dt = \omega - \omega_m.$$

Integrating again with respect to time we get the angular deviation from the mean position, *i.e.*

$$\theta = \int(\omega - \omega_m)\,dt.$$

It will be convenient to evaluate these quantities graphically, beginning with the curve of engine torque plotted against time. It should be observed that over a given recurrent cycle the area bounded by the curve of variable engine torque is equal to that bounded by the line which represents both the mean engine torque and the external load torque. Over the same period there is no permanent change in velocity or position, both velocity and position being the same at intervals equal to the period of the cycle. The changes are temporary, and are caused by the fluctuation of engine torque over load torque, and it is with these alone we are concerned.

The first step is to draw in a mean torque line on the diagram of engine torque to make clear the torque fluctuations.

The first derived curve from which the changes in velocity are deduced is obtained by integrating the areas of the engine torque diagram, treating those above the mean torque line as positive and those below as negative. In the new diagram the height of each ordinate is proportional to the area of the previous diagram up to that point. As the point at which the integration starts is usually quite arbitrary, it is necessary to draw in a horizontal line representing the mean velocity before changes of velocity from the mean can be scaled off.

This may be done by drawing a horizontal line at the mean distance of the ordinates from the axis of time.

The second derived curve, giving the angular deviation, is obtained by integrating the areas of the fluctuations in the

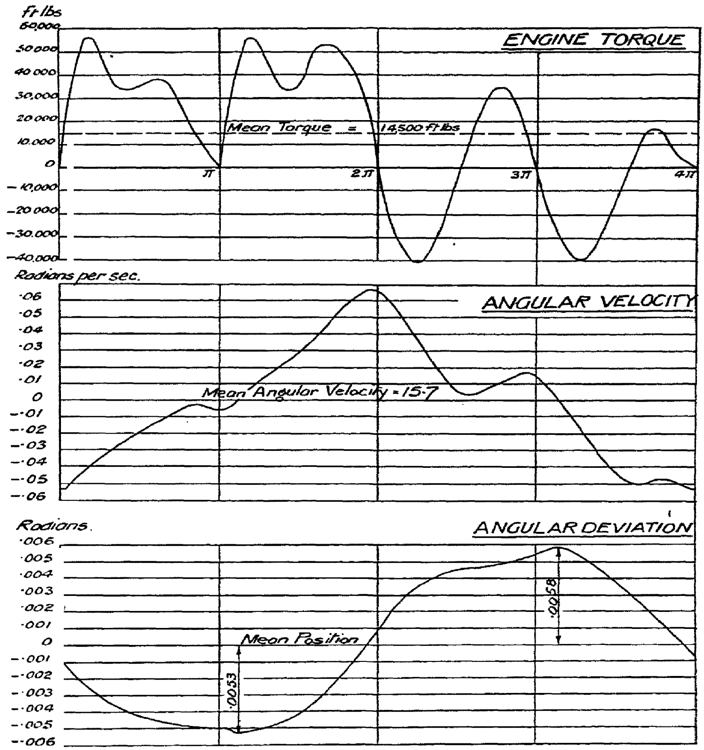


FIG. 41.

velocity curve in the same way. Again, the integration is probably started from an arbitrary point, and before actual deviations from the mean position can be scaled off a mean position line must be drawn in. The maximum change in position is at once apparent, and is proportional to the height

from maximum crest to maximum trough, but it is rarely that the maximum deviation on the one side is equal to the maximum deviation on the other, although the sum of the positive deviations must equal the sum of the negative deviations; hence the necessity for a line of mean position.

A slight error is introduced into the method in its practical application, by the fact that it is impossible to start with a true torque time diagram and that, instead, a torque crank angle diagram is used which can, of course, take no account of the deviations it is the object of this investigation to determine. If strict accuracy is required, the torque crank angle diagram may be used to determine the angular deviations to a first degree of approximation, from which a most nearly true torque time diagram may be constructed and the work repeated. It is, however, safe to say that for the small deviations permissible in electrical work this consideration may be wholly ignored.

To illustrate the application of the method, the following example of a 400 H.-P. "Crossley" gas engine is worked out. The cylinders are arranged *vis-à-vis*, and the connecting rods work on a single crank pin. The torque diagram is shown in Fig. 41. The speed of crank shaft is 150 r.p.m., and the moment of inertia of all rotating parts is 1,160,000 pounds feet ^{2}.

Let ω_1 and ω_2 be minimum and maximum speeds. Then the mean speed $\omega_m = \dfrac{\omega_1 + \omega_2}{2}$.

Let n = fractional fluctuation of speed from ω_1 to ω_2, *i.e.*

$$n = \frac{\omega_2 - \omega_1}{\omega_m} \text{ and } n\omega_m = \omega_2 - \omega_1.$$

The change in kinetic energy of the flywheel from ω_1 to ω_2 is

$$\frac{Mk^2}{2}(\omega_2{}^2 - \omega_1{}^2) = Mk^2(\omega_2 - \omega_1)\frac{(\omega_2 + \omega_1)}{2}$$

$$= Mk^2 n \,.\, \omega^2{}_m.$$

This must equal the fluctuation in energy as obtained by integrating the variations of the torque diagram above and below the mean torque line, between these limits.

It is found to be 68,200 foot-pounds or 68,200 g foot-poundals.

$$Mk^2 = 1{,}160{,}000 \text{ pound feet}^2$$

and $$\omega_m = \frac{2\pi N}{60} = \frac{2\pi \times 150}{60} = 15{\cdot}7 \text{ radians per second.}$$

Equating the fluctuation of energy to the change in kinetic energy of the flywheel between minimum and maximum speed, we have

$$68{,}200\, g = 1{,}160{,}000\, n \, . \, 15{\cdot}7^2.$$

Therefore $$n = \frac{68{,}200\, g\, .}{1{,}160{,}000 \times 15{\cdot}7^2} = \frac{1}{130}.$$

This enables the scale of velocity diagram to be fixed, the vertical distance between the trough and crest corresponding to

$$n\omega_m = \frac{1}{130} \times 15{\cdot}7 = {\cdot}1208 \text{ radian per second.}$$

In order to fix the scale of the angular deviation diagram, it must be remembered that this deviation is caused by the variation from the mean velocity. The total deviation represented by the vertical distance between the trough and crest of the deviation curve will be caused by the corresponding fluctuation in velocity above the mean velocity curve. This portion of the velocity diagram integrated with respect to time works out to ·0111 radian, which gives the total angular deviation and fixes the scale of the deviation diagram.

From the diagram it is seen that during the cycle the flywheel has a maximum deviation of ·0053 radian behind its mean position and a maximum deviation of ·0058 in front of its mean position, or, expressed in degrees of circumference, $-\,{\cdot}304^\circ$ and $+\,{\cdot}332^\circ$.

It will be interesting to compare the results with those obtained from applying the approximate formula of Part I, viz. :—

$$\text{Error in degrees of circumference} = \frac{22{\cdot}5}{N} \, . \, \frac{R_2 - R_1}{R_0}$$

where N = number of impulses per revolution

The case considered is somewhat special in that, although there are actually two impulses in one revolution followed by

no impulses in the next revolution, the impulses follow so rapidly on one another that their effect on the flywheel is practically that of one continuous impulse and should be so regarded.

We have, then,
$$N = \frac{1}{2}$$

and
$$\frac{R_2 - R_1}{R_0} = \frac{1}{130}$$

and Error in degrees of circumference
$$= 2 \times 22{\cdot}5 \times \frac{1}{130}$$
$$= {\cdot}346,$$

which result agrees very well with the true value determined graphically.

APPENDIX

Hyperbolic Functions

The solution of problems in the whirling and vibration of uniform shafts is facilitated by the use of hyperbolic functions.

Corresponding to the trigonometrical formulæ,

$$\begin{aligned}
\sin^2 x &+ \cos^2 x = 1,\\
\sec^2 x &- \tan^2 x = 1,\\
\operatorname{cosec}^2 x &- \cot^2 x = 1,
\end{aligned}$$

we have for the hyperbolic functions,

$$\begin{aligned}
\cosh^2 x &- \sinh^2 x = 1,\\
\operatorname{sech}^2 x &+ \tanh^2 x = 1,\\
\operatorname{cosech}^2 x &- \coth^2 x = -1.
\end{aligned}$$

The exponential values of the trigonometrical functions $\sin x$ and $\cos x$ are as follows:—

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$

$$\cos x = \frac{e^{ix} + e^{-ix}}{2} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}$$

where

$$i = \sqrt{-1} \text{ and } e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots, \text{ etc.}$$
$$= 2{\cdot}7182818 \ldots$$

The corresponding values of the hyperbolic functions $\sinh x$ and $\cosh x$ are:

$$\sinh x = \frac{e^{x} - e^{-x}}{2} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \ldots$$

$$\cosh x = \frac{e^{x} + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \ldots$$

Table of Exponentials and Hyperbolic Functions.

x	e^x	e^{-x}	cosh x	sinh x	tanh x
·1	1·1052	·9048	1·0050	·1002	·0997
·2	1·2214	·8187	1·0201	·2013	·1974
·3	1·3499	·7408	1·0453	·3045	·2913
·4	1·4918	·6703	1·0811	·4108	·3799
·5	1·6487	·6065	1·1276	·5211	·4621
·6	1·8221	·5488	1·1855	·6367	·5370
·7	2·0138	·4966	1·2552	·7586	·6044
·8	2·2255	·4493	1·3374	·8881	·6640
·9	2·4596	·4066	1·4331	1·0265	·7163
1·0	2·7183	·3679	1·5431	1·1752	·7616
1·1	3·0042	·3329	1·6685	1·3357	·8005
1·2	3·3201	·3012	1·8107	1·5095	·8337
1·3	3·6693	·2725	1·9709	1·6984	·8617
1·4	4·0552	·2466	2·1509	1·9043	·8854
1·5	4·4817	·2231	2·3524	2·1293	·9051
1·6	4·9530	·2019	2·5775	2·3756	·9217
1·7	5·4739	·1827	2·8283	2·6456	·9354
1·8	6·0496	·1653	3·1075	2·9422	·9468
1·9	6·6859	·1496	3·4177	3·2682	·9563
2·0	7·3891	·1353	3·7622	3·6269	·9640
2·1	8·1662	·1225	4·1443	4·0219	·9704
2·2	9·0251	·1108	4·5679	4·4571	·9758
2·3	9·9742	·1003	5·0372	4·9370	·9801
2·4	11·0232	·0907	5·5570	5·4662	·9837
2·5	12·1825	·0821	6·1323	6·0502	·9866
2·6	13·4638	·0743	6·7690	6·6947	·9890
2·7	14·8797	·0672	7·4735	7·4063	·9910
2·8	16·4446	·0608	8·2527	8·1919	·9926
2·9	18·1741	·0550	9·1146	9·0596	·9940
3·0	20·0855	·0498	10·068	10·018	·9951
3·1	22·1980	·0450	11·122	11·076	·9959
3·2	24·5325	·0408	12·287	12·246	·9967
3·3	27·1126	·0369	13·575	13·538	·9973
3·4	29·9641	·0334	14·999	14·965	·9978
3·5	33·1155	·0302	16·573	16·543	·9982
3·6	36·5982	·0273	18·313	18·285	·9985
3·7	40·4473	·0247	20·236	20·211	·9988
3·8	44·7012	·0224	22·362	22·339	·9990
3·9	49·4024	·0202	24·711	24·691	·9992
4·0	54·5982	·0183	27·308	27·290	·9993
4·1	60·3403	·0166	30·178	30·162	·9995
4·2	66·6863	·0150	33·351	33·336	·9996
4·3	73·6998	·0136	36·857	36·843	·9996
4·4	81·4509	·0123	40·732	40·719	·9997
4·5	90·0171	·0111	45·014	45·003	·9997
4·6	99·4843	·0101	49·747	49·737	·9998
4·7	109·9472	·0091	54·978	54·969	·9998
4·8	121·5104	·0082	60·759	60·751	·9999
4·9	134·2898	·0074	67·149	67·141	·9999
5·0	148·4132	·0067	74·210	74·203	·9999

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